MAYA

"SLOPE" IN SOLID ANALYTIC GEOMETRY

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1. Introduction. For some time it has concerned the author that the usual treatment of the analytic geometry of lines in space is not an immediate generalization of their treatment in the plane. The particular point at issue is the use of "slope" in the plane and of "direction cosines" in space. It is unsound pedagogy to treat two dimensional concepts and three dimensional concepts as separate entities when they are in fact two special cases of a general theory. This situation has been recognized by others and has been overcome in several recent texts which use direction cosines in the plane as well as in space. From the purely geometrical point of view this is ideal; but it raises another difficulty when the analytic geometry is applied to calculus. For slope is essential to the calculus as it is usually presented, and many courses in analytic geometry may not have the time to include discussions of both direction cosines and slope.

It is not generally recognized that the notion of slope can be extended to solid analytic geometry, and the purpose of this paper is to show how this can be accomplished. The use of slope in both the plane and space has the advantages: (1) it provides a unified treatment of plane and solid analytic geometry, (2) it provides a direct approach to the derivative in the plane and to partial derivatives in space. It has the disadvantage of lack of symmetry in the variables, thus losing elegance, and further it suffers from the need of considering special cases (usually glossed over in the plane) which arise from the infinite values introduced by the use of the tangent function. These disadvantages are serious, but the author feels that the advantages are sufficiently strong for this approach to merit the attention of teachers of college mathematics.

2. Slopes of lines. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two points on a line, l; then the slopes p and q of l are given by the definition:

DEFINITION.

$$\text{The } x\text{-slope, } p, \text{ of } l \text{ is } p = \frac{z_2 - z_1}{x_2 - x_1}$$

$$\text{The } y\text{-slope, } q, \text{ of } l \text{ is } q = \frac{z_2 - z_1}{y_2 - y_1}.$$

(The reader will note that p corresponds to $\partial z/\partial x$ and q to $\partial z/\partial y$ as is customary in treatments of partial differential equations.)

It is clear that p is the ordinary (two-dimensional) slope of the projection of l in the XZ plane, and similarly that q is the slope of the projection of l in the YZ plane. It follows, then, from plane analytic geometry that p and q are independent of the two points chosen to define them. If l is parallel to the YZ plane, $x_2 - x_1 = 0$; in this case we say that $1/p = 0$, or loosely that $p = \infty$. Similarly when l is parallel to the XZ plane, $y_2 - y_1 = 0$ and $1/q = 0$ or $q = \infty$.

Occasionally it is useful to consider the slope of the projection of l into the XY plane. The slope of this projection will be called the xy-slope of l and is given by the definition:

DEFINITION.

$$\text{The } xy\text{-slope, } m, \text{ of } l \text{ is } m = \frac{y_2 - y_1}{x_2 - x_1}.$$

For lines parallel to the XY plane it follows that $p=0$ and $q=0$, and m is the usual slope of a line as defined in plane geometry. For lines parallel to the YZ plane, $1/m=0$ or $m=\infty$.

It is useful to point out that:

For lines parallel to the X axis: $p=0$, q is undefined $(=0/0)$. $m=0$.

For lines parallel to the Y axis: p is undefined $(=0/0)$, $q=0$, $m=\infty$.

For lines parallel to the Z axis: $p=\infty$, $q=\infty$, m is undefined $(=0/0)$.

It further follows that p/q is defined except for lines parallel to the XY plane, and that unless p/q is undefined: $m=p/q$. This shows that except for the special case just mentioned, it is sufficient to deal with p and q exclusively and to omit reference to m.

3. The equations of a line. In this section the equations of a line not parallel to the XY plane will be developed first. Such a line will be called a general line. Lines parallel to the XY plane will be called special lines.

Point-slope Equations. To find the equations of a general line given a point on it, say (x_1, y_1, z_1) and its slopes p and q, let (x, y, z) be an arbitrary point on the line. Then if $1/p \neq 0$ and $1/q \neq 0$:

$$\frac{z - z_1}{x - x_1} = p; \qquad \frac{z - z_1}{y - y_1} = q$$

or

$$z - z_1 = p(x - x_1); \qquad z - z_1 = q(y - y_1)$$

are the equations of the line. If $1/p=0$, the first of these equations becomes $x-x_1=0$; if $1/q=0$, the second of these equations becomes $y-y_1=0$.

Two-point Equations. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two given points, we seek to find the equations of the line defined by these two points. By substituting the values of p and q given by their definitions into the point-slope equations of the line, we have the desired result:

When $x_2-x_1 \neq 0$ and $y_2-y_1 \neq 0$:

$$z - z_1 = \frac{z_2 - z_1}{x_2 - x_1}(x - x_1); \qquad z - z_1 = \frac{z_2 - z_1}{y_2 - y_1}(y - y_1).$$

When $x_2-x_1=0$, the first of these equations becomes $x=x_1$. When $y_2-y_1=0$, the second of these equations becomes $y=y_1$. All of these possibilities are included by writing the equations in the form:

$(z - z_1)(x_2 - x_1) = (z_2 - z_1)(x - x_1)$; $(z - z_1)(y_2 - y_1) = (z_2 - z_1)(y - y_1)$.

Special lines. A special line is one parallel to the XY plane. Its equations follow from plane geometry:

Point-slope form: $z = z_1$; $y - y_1 = m(x - x_1)$

Two-point form: $z = z_1$; $(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1)$.

4. Condition that two lines be perpendicular. We consider first two general lines. These lines will be perpendicular if the lines l_1 and l_2 passing through the origin and parallel to them are perpendicular. If the slopes of the given lines are p_1, q_1 and p_2, q_2 respectively, it follows that l_1 and l_2 have these same slopes since parallel lines have equal slopes. Since the lines are general, none of the quantities p_1, q_1, p_2, q_2 are zero. It is easy to show that the points $P_1(1/p_1, 1/q_1, 1)$ and $P_2(1/p_2, 1/q_2, 1)$ lie on l_1 and l_2 respectively. We do not exclude the cases where any of $1/p_1$, $1/q_1$, $1/p_2$, $1/q_2$ are zero. In order that l_1 be perpendicular to l_2 it is necessary that the triangle P_1OP_2 have a right angle at O. By applying Pythagoras' Theorem, we find the relation:

$$1/p_1p_2 + 1/q_1q_2 + 1 = 0.$$

This result is readily seen to be an appropriate generalization of the condition for the perpendicularity of two lines in a plane.

If either l_1 or l_2 is a special line, the condition for perpendicularity is $m_1m_2 = -1$ provided that neither m_1 nor m_2 is ∞. If $m_1 = \infty$, m_2 must be zero; and if $m_2 = \infty$, m_1 must be zero.

More generally, to find the angle, θ, between two general lines, we apply the law of cosines to the triangle P_1OP_2, the angle θ being at the vertex O. The result is:

$$\cos\theta = \frac{\dfrac{1}{p_1p_2} + \dfrac{1}{q_1q_2} + 1}{\sqrt{\dfrac{1}{p_1^2} + \dfrac{1}{q_1^2} + 1}\sqrt{\dfrac{1}{p_2^2} + \dfrac{1}{q_2^2} + 1}}.$$

5. The slopes of planes. For our purposes planes will be divided into two classes as follows: planes parallel to the Z axis will be called special planes; other planes are general planes. A general plane, π, will intersect the XZ plane and the YZ plane in two lines called the traces of π, whose slopes in these planes are p and q respectively. A plane, π, then has two slopes which are defined as follows:

DEFINITION. The x-slope, P, of π is the slope, p, of its trace in the XZ plane. The y-slope, Q, of π is the slope, q, of its trace in the YZ plane.

These slopes are always defined for general planes, although in certain cases they may be infinite.

It is useful to introduce the xy-slope of a plane as follows:

DEFINITION. The xy-slope, M, of π is the slope, m, of its trace in the XY plane.

This slope will seldom be used except for special planes.

The slopes of certain planes have particular values which are of some interest:

A plane parallel to the XY plane has: $P=0$, $Q=0$, and M is undefined.
A plane parallel to the XZ plane has: P is undefined, $Q=\infty$, and $M=0$.
A plane parallel to the YZ plane has: $P=\infty$, Q is undefined, and $M=\infty$.

6. The equations of a plane. It is proved in standard texts that every equation of the first degree: $Ax+By+Cz+D=0$ is the equation of a plane, and every plane has an equation of this form.

Three-point Form. Let $P_1(x_1, y_1, z_1)$; $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ be any three non-collinear points. Also let:

$$T_{yz} = 1/2 \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}; \quad T_{zx} = 1/2 \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix};$$

$$T_{xy} = 1/2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & x_3 & 1 \end{vmatrix}; \quad T_{xyz} = 1/2 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

It will be noted that T_{yz}, T_{zx}, T_{xy} are the areas (with a specified algebraic sign) of the projections of the triangle $P_1P_2P_3$ upon the YZ plane, the XZ plane, and the XY plane respectively. Not all of these are simultaneously zero. Then it is proved in many standard texts that the plane, π, passing through these three points has the equation:

$$T_{yz}x + T_{zx}y + T_{xy}z - T_{xyz} = 0.$$

This is the three-point form of the equation of a plane.

Slopes of a Plane in Terms of Three Non-collinear Points on it. The trace of π in the XZ plane is obtained by putting $y=0$ in the equation just above. The x-slope, P, of the plane equals the x-slope, p, of this trace. Therefore: $P=p=-T_{yz}/T_{xy}$. Similarly $Q=-T_{zx}/T_{xy}$; and $M=-T_{yz}/T_{zx}$. The analogy with the slope of a line in plane geometry is now clear. For if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points on a line, its slope is $m=y_2-y_1/x_2-x_1$. Here y_2-y_1 is the length of the projection of the segment P_2P_1 on the Y axis and x_2-x_1 is the length of the projection of P_2P_1 on the X axis. The slope is then the ratio of the lengths of these projections.

Reference should be made to section 5 for a statement of the cases in which P, Q, and M are not defined. Unless P/Q is undefined it follows that $M=-P/Q$; and consequently we can deal exclusively with P and Q except for this special case.

Slope, Z-intercept Form. For general planes $T_{xy} \neq 0$ and hence the three-point form of the equation of a plane may be solved for z (taking into account the values for P and Q) as follows:

$$z = Px + Qy + T_{xyz}/T_{xy}.$$

The last term of this equation does not involve any unknown and is easily seen to equal the Z-intercept, c, of the plane. Hence we write the equation in the form:

$$z = Px + Qy + c.$$

Special planes do not have an equation of this form.

This result enables us to find the slopes of a general plane as well as its Z-intercept from its equation. The rule is: solve the equation for z; the resulting coefficient of x is the x-slope, P; the coefficient of y is the y-slope, Q, and the constant term is the Z-intercept, c. The reader will note the similarity of this procedure to that used for finding the slope and Y-intercept of a line in the plane.

Point-slope Form. Given the slopes of a plane and a point (x_1, y_1, z_1) on it we seek to find its equation. The slope, Z-intercept equation is

$$z = Px + Qy + c.$$

Since the point lies on this plane: $z_1 = Px_1 + Qy_1 + c$. Subtracting these two equations we obtain the desired result:

$$z - z_1 = P(x - x_1) + Q(y - y_1).$$

If a special plane is not parallel to the YZ plane, its equation is readily seen to be:

$$y - y_1 = M(x - x_1).$$

A plane parallel to the YZ plane has the equation $x = x_1$.

7. Conditions that a plane and a line be perpendicular. It is an elementary result that a line is perpendicular to a plane if and only if it is perpendicular to two non-parallel lines in that plane. If π is a general plane, its trace in the XZ plane has slopes: $p = P$ and $1/q = 0$; and its trace in the YZ plane has the slopes $1/p = 0$ and $q = Q$. A general line l whose slopes are p and q is perpendicular to π provided it is perpendicular to these two traces. The conditions follow immediately from the results of section 4, namely:

$$\frac{1}{pP} + 1 = 0; \qquad \frac{1}{qQ} + 1 = 0.$$

Or: A general line is perpendicular to a general plane if and only if the slopes of the line are the negative reciprocals of the slopes of the plane. This may also be worded: the slopes of the normals to a general plane are the negative reciprocals of those of the plane.

In the interests of brevity the treatment of special lines and planes is omitted. The chief facts are: (1) A special line is never perpendicular to a general plane,

a special plane is never perpendicular to a general line; (2) A special line is perpendicular to a special plane when $mM=-1$.

8. The distance from a plane to a point. Let π be the general plane $Ax+By+Cz+D=0$ $(C\neq 0)$ and $R(x_1, y_1, z_1)$ be any point. The slopes of π are $P=-A/C$ and $Q=-B/C$. The normal to π therefore has the slopes: $p=C/A$ and $q=C/B$. So the equations of the normal to π passing through R are:

$$z-z_1=C/A(x-x_1); \qquad z-z_1=C/B(y-y_1).$$

To find the coordinates of the intersection of this normal with π, we first write the above equations in the form:

$$x=x_1+A/C(z-z_1); \qquad y=y_1+B/C(z-z_1).$$

Since $C\neq 0$, these equations are always possible. Substituting these values in the equation of the plane and solving for z, we have the result:

$$z=z_1-C\left(\frac{Ax_1+By_1+Cz_1+D}{A^2+B^2+C^2}\right).$$

The value of z thus determined is the z coordinate of the point S which is the foot of the perpendicular to π from R. The other coordinates of S are found to be:

$$y=y_1-B\left(\frac{Ax_1+By_1+Cz_1+D}{A^2+B^2+C^2}\right); \quad \text{and}$$

$$x=x_1-A\left(\frac{Ax_1+By_1+Cz_1+D}{A^2+B^2+C^2}\right).$$

The distance, d, from π to R is the distance SR and hence:

$$d=\frac{Ax_1+By_1+Cz_1+D}{\pm\sqrt{A^2+B^2+C^2}}.$$

According to the customary convention, the sign of the radical is chosen opposite to that of D. This result can be extended without difficulty to special planes.

9. Application to partial differentiation. Let us consider a surface $z=f(x, y)$ and a point R (x_1, y_1, z_1) on it. Let S be the point $(x_1+\Delta x, y_1, z_1+\Delta z)$ where $z_1+\Delta z=f(x_1+\Delta x, y_1)$. And let T be the point $(x_1, y_1+\delta y, z_1+\delta z)$ where $z_1+\delta z=f(x_1, y_1+\delta y)$. The plane passing through R, S, and T has the slopes: $P=\Delta z/\Delta x$ and $Q=\delta z/\delta y$. Thus the equation of this plane is:

$$z-z_1=\frac{\Delta z}{\Delta x}(x-x_1)+\frac{\delta z}{\delta y}(y-y_1).$$

The tangent plane to the surface at R is defined to be the limiting position of the RST plane when $\Delta x \to 0$ and $\delta y \to 0$ independently. The slopes of the limit plane are given by

$$P = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial x} \quad \text{and} \quad Q = \lim_{\delta y \to 0} \frac{\delta z}{\partial y} = \frac{\partial z}{\partial y}.$$

Thus the geometrical interpretation of $\partial z/\partial x$ and $\partial z/\partial y$ is that they are the slopes of the tangent plane to the surface at the given point. In this fashion partial derivatives may be introduced by a geometrical argument parallel to that used for ordinary derivatives.

Another, less conventional, approach to partial derivatives is the following. Let the surface and the point R be defined as above. But now let S be the point $(x_1+\Delta x,\ y_1+\Delta y,\ z_1+\Delta z)$ where $z_1+\Delta z=f(x_1+\Delta x,\ y_1+\Delta y)$; and let T be the point $(x_1+\delta x,\ y_1+\delta y,\ z_1+\delta z)$ where $z_1+\delta z=f(x_1+\delta x,\ y_1+\delta y)$. From the formulas given in section 6 it follows that the slopes of the plane through R, S, and T are as follows:

$$P = \frac{\Delta y \delta z - \Delta z \delta y}{\Delta y \delta x - \Delta x \delta y}; \quad \text{and} \quad Q = \frac{\Delta z \delta x - \Delta x \delta z}{\Delta y \delta x - \Delta x \delta y}.$$

The limits of these quantities as S and T approach R are then the slopes of the tangent plane, and are equal to the corresponding partial derivatives. Certain restrictions, however, must be placed on this limit process, for there are cases for which the limiting position of the plane RST is not the tangent plane. For example, let us suppose that R, S, and T all lie on a circular cross section of a right circular cylinder. If S and T approach R along arcs of this circle, the limit plane is a normal plane! A restriction sufficient to avoid this difficulty is the following. Let S approach R along a differentiable curve C_1, and let T approach R along a differentiable curve C_2. The limiting position of the plane RST will be a tangent plane if the curves C_1 and C_2 have distinct tangent lines at R.

We recall that the numerators and denominators of the above expressions for P and Q are equal to twice the signed areas of the projections of the triangle RST upon the corresponding coordinate planes. The partial derivatives are therefore equal to the limits of the ratios of the areas of the appropriate pairs of these projections. It will be noticed that this interpretation is particularly useful in establishing the formula for the element of surface area of the surface $z=f(x, y)$:

$$d\sigma = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dy dx.$$

ON SETS OF DISTANCES OF n POINTS

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1. The function f(n). Let $[P_n]$ be the class of all planar subsets P_n of n points and denote by $f(n)$ the minimum number of different distances determined by its n points for P_n an element of $\{P_n\}$. Clearly, $f(3)=1$ (with the three points forming the vertices of an equilateral triangle) $f(4)=2$, $f(5)=2$. The following theorem establishes rough bounds for arbitrary n. Though I have sought to improve this result for many years, I have not been able to do so.

THEOREM 1. *The minimum number $f(n)$ of distances determined by n points of a plane satisfies the inequalities*

$$(n-3/4)^{1/2}-1/2 \leqq f(n) \leqq cn/(\log n)^{1/2}.$$

Proof. Let P_1 be an arbitrary vertex of the least convex polygon determined by the n points, and denote by K the number of different distances occurring among the distances P_1P_i $(i=2, 3, \cdots, n)$. If N is the maximum number of times the same distance occurs, then clearly $KN \geqq n-1$.

If r is a distance that occurs N times then there are N points on the circle with center P_1 and radius r, which all lie on the same semi-circle (since P_1 is a vertex of the least convex polygon). Denoting these points by $Q_1, Q_2, \cdots, Q_N$, we have $Q_1Q_2<Q_1Q_3<\cdots<Q_1Q_N$, and these $N-1$ distances are pairwise distinct. Thus $f(n) \geqq \max(N-1, (n-1)/N)$, which is a minimum when $N(N-1)=n-1$. This yields the first part of the theorem.

Considering now the points (x, y) with *integer* coordinates for $0 \leqq x, y \leqq n^{1/2}$, we obtain at least n points P_i which pairwise have distances of the form $(u^2+v^2)^{1/2}$, $0 \leqq u \leqq n^{1/2}$, $0 \leqq v \leqq n^{1/2}$. Now it is well-known that the number of different integers not exceeding $2n$ which are of the form u^2+v^2 is less than $cn/(\log n)^{1/2}$, and the proof is complete.*

For n points in k-dimensional space the same method yields $c_1 n^{1/k} < f(n) < c_2 n^{2/k}$.

2. Some conjectures concerning f(n). Let us assume that our n points form a convex polygon. Then I conjecture that $f(n) \geqq [n/2]$, with the equality sign valid when the n points are vertices of a regular n-gon. I am unfortunately unable to prove this. The following conjecture is stronger: In every convex polygon there is at least one vertex with the property that no three vertices of the polygon are equally distant from it. If this is the case, then clearly we would obtain $[n/2]$ different distances by considering all the distances from such a vertex.

A still stronger conjecture is that on every convex curve there exists a point P such that every circle with center P intersects the curve in at most 2 points.

3. The function g(n; r). Denoting by $g(n; r)$ the maximum number of times a given distance r can occur among n points of a plane we establish

* Landau, Verteilung der Primzahlen, vol. 2.

THEOREM 2. $n^{1+c/\log\log n} < g(n;r) < n^{3/2}$.

Proof. Assuming that there are x_i points at distance r from P_i, clearly $g(n;r) = \max \frac{1}{2}\sum_{i=1}^{n} x_i$. We suppose that $x_1 \geqq x_2 \geqq \cdots \geqq x_n$. Now the x_i points at distance r from P_i can contain at most two points with distance r from P_j. Hence

$$\sum_{i=1}^{j} (x_i - 2i + 2) \leqq n \quad \text{for} \quad j = 1, 2, \cdots, n. \tag{1}$$

Put $[n^{1/2}] = a$, $n^{1/2} - a = \epsilon$, $0 \leqq \epsilon < 1$. We have from (1)

$$\begin{aligned} x_1 + x_2 + \cdots + x_a &\leqq n + 2\binom{a}{2} = 2n - 2\epsilon n^{1/2} + \epsilon^2 - n^{1/2} + \epsilon \\ &< 2n - 2\epsilon n^{1/2} \end{aligned} \tag{2}$$

for $n \geqq 4$. Thus

$$x_a < \frac{1}{a}(2n - 2\epsilon n^{1/2}) = 2n^{1/2}. \tag{3}$$

Hence from (2) and (3)

$$\sum_{i=1}^{n} x_i < 2n - 2\epsilon n^{1/2} + (n - a)2n^{1/2} = 2n^{3/2}$$

or

$$g(n; r) < n^{3/2}.$$

By again considering the set of points (x, y), $0 \leqq x, y \leqq a$ we easily obtain (using well known theorems about the number of solutions of $u^2 + v^2 = m$)*

$$g(n) > n^{1+c/\log\log n}$$

which completes the proof.

It seems likely that $g(n) < n^{1+\epsilon}$.

4. Maximum and minimum distances. If r is the diameter of the points P_i, it is well known that r can occur only n times.† This follows almost immediately from the fact that if $\overline{P_1P_2} = r$ and $\overline{P_3P_4} = r$ the lines P_1P_2 and P_3P_4 must intersect, for otherwise a simple argument shows that the diameter of $P_1P_2P_3P_4$ would be greater than r. Connect P_i with P_j if and only if their distance is r. We distinguish two cases. In Case 1, every P_i is connected with at most two other P's. In this case the number of lines, *i.e.*, of pairs of points at distance r is clearly $\leqq n$.

* See *e.g.* P. Erdös, London Math. Soc. Journal, 1937, vol. 12, p. 133. The proof would depend on the prime number theorem for primes of the form $4k+1$ (or on some weaker elementary result concerning the distribution of primes of the form $4k+1$).

† Jahresbericht der Deutschen Math. Vereinigung, vol. 43, 1934, p. 114.

If P_1 would be connected with three vertices say P_2, P_3, P_4 where P_1P_3 is between P_1P_2 and P_1P_4 then P_3 can not be connected with any other P_i, since P_3P_i would have to intersect both P_1P_2 and P_1P_4 (the angle $P_2P_1P_4$ is of course $\leqq\pi/3$), and thus be greater than r. Now we can just omit P_3 and since both the number of points and the number of distances are reduced by 1, the proof can be completed by induction.

It would be interesting to have an analogous result for n points in k dimensional space. Vázsonyi* conjectured that in three-dimensional space the maximum distance can not occur more than $2n-2$ times.

If one could prove that in k-dimensional space the maximum distance can not occur more than kn times, the following conjecture of Borsuk would be established: *Each k-dimensional* subset of diameter 1 can be decomposed into $k+1$ summands each having diameter <1.

Let now r' denote the minimal distance between any two P's. First it is easy to see that r' can not occur more often than $3n$ times. This is immediately clear from the fact that since r' was the minimal distance between any two P's, there can be no more than 6 P's at distance r' from any given P.

Connect P_i with P_j if and only if their distance is r'. A simple argument shows that no two such lines P_1P_2 and P_3P_4 can intersect (otherwise there would be two P's at distance $<r'$). Thus the graph we obtain is planar, and from Euler's theorem it follows that the number of edges of such a graph is not greater than $3n-6$. Thus we have proved the following

THEOREM 3. *Let the maximum and minimum distances determined by n points in a plane be denoted by r and r', respectively. Then r can occur at most n times and r' at most $3n-6$ times.*

It is easy to give n points where the maximum distance occurs exactly n times. By more complicated arguments we can prove that the minimal distance r' can occur not more than $3n-cn^{1/2}$ times, where c is a constant. On the other hand the example of the triangular lattice shows that r' can occur $3n-c_1n^{1/2}$ times. I did not succeed in determining exactly how often r' can occur.

One could try to generalize Theorem 3 to higher dimensions. But already the case of three-dimensional space presents great difficulties. It would be of some interest to determine the maximum number of points on the unit sphere of k dimensions such that the distance of any two is $\geqq 1$.

* Oral communication.

SUMMABILITY OF POWER SERIES

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1. Introduction. Let b_{nk} denote a matrix of real or complex numbers. This matrix determines a series-to-sequence transformation B,

$$\sigma_n = \sum_{k=0}^{\infty} b_{nk}u_k, \tag{1}$$

by means of which a given series $u_0+u_1+\cdots$ is said to be *summable* to the value σ if the series in (1) converge and define numbers $\sigma_0, \sigma_1, \cdots$ such that

$$\lim_{n\to\infty} \sigma_n = \sigma. \tag{2}$$

The transformation B is said to be *regular* if each convergent series $\sum u_n$ is summable B to the value to which it converges.

We present four theorems characterizing the matrices b_{nk} for which the transformation B has stated properties. Theorem 1 is widely known and we shall give references and historical remarks. In particular, we shall point out that Harald Bohr originated the theorem. Theorem 2 is known. It is believed that Theorems 3 and 4 are new. Finally, in §8, it is shown that a theorem due to I. Schur [1918] is a corollary of Theorem 4. Here and hereafter, the name of an author followed by a date in square brackets indicates that the reference is to be found in the bibliography at the end of this paper.

2. Regularity. The following theorem gives the test which one applies to a matrix b_{nk} to decide whether it determines a regular series-to-sequence transformation.

THEOREM 1. *In order that the series-to-sequence transformation B be regular, it is necessary and sufficient that*

$$\lim_{n\to\infty} b_{nk} = 1 \qquad k = 0, 1, 2, \cdots \tag{3}$$

and there exist a constant M such that

$$\sum_{k=0}^{\infty} |b_{nk} - b_{n,k+1}| < M \qquad n = 0, 1, 2, \cdots. \tag{4}$$

In case $b_{nk}=0$ when $k>n$, the nonvanishing elements of the matrix

$$\begin{array}{lll} b_{00} & b_{01} & b_{02}\cdots \\ b_{10} & b_{11} & b_{12}\cdots \\ b_{20} & b_{21} & b_{22}\cdots \\ \cdots & & \end{array} \tag{5}$$

lie in a triangle and the matrix is said to be triangular. Harald Bohr [1909]

proved that if b_{nk} is a triangular matrix satisfying (3) and (4), then B is regular. This pioneering paper has never received the attention and credit it deserves. The author has never seen a reference to it in the literature of the theory of summability; he first learned of the existence of the paper through a conversation with Bohr in 1931 when the author was a Fellow at Princeton. This paper of Bohr antedates both the work of Silverman [1913], a thesis written at the University of Missouri under the direction of E. R. Hedrick and accepted by the Graduate Faculty in May 1910, and of Toeplitz [1911]. Silverman proved sufficiency, and Toeplitz proved both necessity and sufficiency, of the well known Silverman-Toeplitz conditions that a matrix a_{nk} be such that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}s_k = \lim_{n\to\infty} s_n \tag{6}$$

for each sequence s_n for which $\lim s_n$ exists. For early expositions of the Silverman-Toeplitz theory, see Carmichael [1919] and Hurwitz [1922].

The Silverman-Toeplitz theory was widely publicized for several years before serious attention was given to the series-to-sequence transformation (1). Carmichael [1919] proved sufficiency of the conditions in Theorem 1, and indicated on page 120 that both he and T. H. Hildebrandt then knew that the conditions are necessary as well as sufficient. Perron [1920] proved sufficiency of the conditions. It seems that Hahn [1922] and Takenaka [1922] were the first to publish complete proofs of Theorem 1.

The remainder of this section is inserted in response to the suggestion of a referee that further discussion and a simple example be presented for readers unfamiliar with the theory of summability. A method of summability for series is, by definition, a method for assigning values to series. *Convergence* is the standard example. A series $u_0+u_1+u_2+\cdots$ with partial sums $s_0=u_0$, $s_1=u_0+u_1$, $s_2=u_0+u_1+u_2, \cdots$ is *convergent* (or is *summable by the method of convergence*) to s if $s_n\to\infty$ as $n\to\infty$. Another simple and useful method is the method known as the *arithmetic mean method* and as the *Cesaro method* C_1 *of order* 1. A series $u_0+u_1+\cdots$ is *summable* C_1 to σ if $\sigma_n\to\sigma$ as $n\to\infty$ when

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$

The series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

is not convergent because its sequence 1, 0, 1, 0, 1, 0, $\cdots$ of partial sums does not have a limit. But the C_1 transform

$$1, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{3}{6}, \frac{4}{7}, \frac{4}{8}, \cdots$$

has the limit $\frac{1}{2}$ and accordingly the series is summable C_1 to $\frac{1}{2}$. This illustrates

the fact that series are summable (evaluable) by methods other than convergence.

Writing the C_1 transformation in the form

$$\sigma_n = \sum_{k=0}^{n} \frac{1}{n+1} s_k,$$

one sees that it has the form

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k$$

where, for each $n=0, 1, \cdots$, $a_{nk}=1/(n+1)$ when $0 \leqq k \leqq n$ and $a_{nk}=0$ when $k>n$. Replacing s_0 by u_0, s_1 by u_0+u_1, $\cdots$ in the above sequence-to-sequence C_1 transformation and simplifying the result gives the C_1 series-to-sequence transformation

$$\sigma_n = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) u_k.$$

This has the form (1) where, for each $n=0, 1, 2, \cdots$

$$b_{nk} = \left(1 - \frac{k}{n+1}\right) \qquad 0 \leqq k \leqq n$$

and $b_{nk}=0$ when $k>n$.

3. Summability of absolutely convergent series. We shall abbreviate the statement that a series $\sum u_n$ is summable by a method B by saying that $B\{\sum u_n\}$ exists. The statement that $\sum u_n$ is summable B to the value σ will be abbreviated to $B\{\sum u_n\}=\sigma$.

Theorem 1 characterizes the matrices b_{nk} such that $B\{\sum u_n\}=\sum u_n$ whenever $\sum u_n$ converges; these matrices are the regular matrices. Since each absolutely convergent series is convergent, it is of course true that each regular matrix is such that $B\{\sum u_n\}=\sum u_n$ whenever $\sum u_n$ converges absolutely. Since there are nonabsolutely (conditionally) convergent series, one must admit the possibility that there may be nonregular matrices b_{nk} such that $B\{\sum u_n\}=\sum u_n$ whenever $\sum u_n$ converges absolutely. In fact, when one decreases the class of series for which the formula $B\{\sum u_n\}=\sum u_n$ is required to hold, one has the problem of determining whether this enlarges the class of methods B having the required property.

If B^* is the particular matrix with elements b^*_{nk} defined by

$$b^*_{nk} = \cos \frac{k}{n+1} \qquad n, k = 0, 1, 2, \cdots, \tag{7}$$

the oscillatory character of the cosine implies that (4) fails to hold and hence that B^* is not regular. However the matrix of B^* satisfies the conditions (8) and (9) below; hence, by Theorem 2, $B^*\{\sum u_n\}=\sum u_n$ whenever $\sum u_n$ converges absolutely.

THEOREM 2. *In order that the series-to-sequence transformation B be such that $B\{\sum u_n\} = \sum u_n$ whenever $\sum u_n$ converges absolutely, it is necessary and sufficient that*

$$\lim_{n\to\infty} b_{nk} = 1 \qquad k = 0, 1, 2, \cdots \tag{8}$$

and there exists a constant M such that

$$|b_{nk}| \leqq M \qquad n, k = 0, 1, 2, \cdots . \tag{9}$$

Observe that if one modifies the statement of Theorem 2 by deleting the word absolutely and replacing (9) by (4), one obtains another way of phrasing the content of Theorem 1. Theorem 2 and several related theorems were first proved by Hahn [1922]; they have been rediscovered several times by other authors who apparently were unaware of the broad scope of Hahn's work.

4. Summability of a power series inside the circle of convergence. It is a fundamental fact of the theory of power series that $\sum c_n z^n$ converges absolutely for each z_0 lying inside the circle of convergence. If we set $u_n = c_n z_0^n$, the statement that z_0 lies inside the circle of convergence of $\sum c_n z^n$ becomes equivalent to the statement that $\sum u_n z^n$ has radius of convergence greater than 1. Moreover, the question whether $B\{\sum a_n z^n\}$ exists whenever z lies inside the circle of convergence of $\sum a_n z^n$ becomes equivalent to the question whether $B\{\sum u_n\}$ exists whenever $\sum u_n z^n$ has radius of convergence greater than 1. We select the latter formulation of the question, and characterize in the next theorem (case $R=1$) the class of transformations B for which $B\{\sum u_n\}$ exists whenever $\sum u_n z^n$ has radius of convergence greater than 1. The class is larger than the class for which $B\{\sum u_n\}$ exists whenever $\sum u_n$ converges absolutely.

THEOREM 3. *Let $R \geqq 0$. In order that the series-to-sequence transformation B be such that $B\{\sum u_n\}$ exists whenever the power series $\sum u_n z^n$ has radius of convergence greater than R, it is necessary and sufficient that there exist constants $\beta_0, \beta_1, \cdots$ such that*

$$\lim_{n\to\infty} b_{nk} = \beta_k \qquad k = 0, 1, 2, \cdots \tag{10}$$

and, corresponding to each number r for which $r > R$, a constant $M(r)$ such that

$$|b_{nk}| \leqq M(r) r^k \qquad n, k = 0, 1, \cdots . \tag{11}$$

If (10) and (11) hold and $\sum u_n z^n$ has radius of convergence greater than R, then $\sum \beta_k u_k$ converges absolutely and

$$B\{\sum u_k\} = \beta_0 u_0 + \beta_1 u_1 + \cdots . \tag{12}$$

5. Proof of necessity of the conditions in Theorem 3. Let (1) be a transformation such that $B\{\sum u_n\}$ exists whenever $\sum u_n z^n$ has radius of convergence greater than R. Proof of necessity of (10) is very easy. Let q be a nonnegative integer and let $\sum u_n$ be the particular series for which $u_q = 1$ and $u_n = 0$ when

$n \neq q$. Then $\sum u_n z^n$ has radius of convergence greater than R and accordingly the transform, which is b_{nq}, must be convergent to a limit which we may call β_q. Thus $b_{nq} \to \beta_q$ as $n \to \infty$, and we have proved necessity of (10).

A condition of the nature of (11), but weaker than (11), is obtained very easily. Let r be a fixed number such that $r > R$. Since $\sum r^{-n} z^n$ has radius of convergence $r > R$, the transform

$$\sigma_n = \sum_{k=0}^{\infty} b_{nk} r^{-k} \tag{13}$$

must exist, that is, the series on the right must be convergent for each $n = 0, 1, 2, \cdots$. Since, for each n, the series $\sum b_{nk} r^{-k}$ must be convergent, the terms of the series must be bounded. Thus there is a constant $M_n(r)$, depending on n and r, such that

$$| b_{nk} r^{-k} | \leqq M_n(r) \qquad k = 0, 1, 2, \cdots . \tag{14}$$

We now suppose, intending to establish a contradiction of our hypothesis, that the stronger condition (11) does not hold. Then, for some r_1 for which $r_1 > R$, there is no constant M such that $| b_{nk} r_1^{-k} | \leqq M$ for each pair of integers n and k. From (10) it follows that, for each fixed k, the numbers $| b_{nk} r_1^{-k} |$ form a bounded sequence; and from (14) it follows that, for each fixed n, the numbers $| b_{nk} r_1^{-k} |$ form a bounded sequence. Hence the numbers $| b_{nk} r_1^{-k} |$ form an unbounded double sequence of which each row is bounded and each column is bounded. Henceforth, when α, β and γ are symbols involving subscripts, we write $b(\alpha, \beta)$ for $b_{\alpha\beta}$ and $r_1(\gamma)$ for $r_1^{-\gamma}$. Let $q_1, q_2, \cdots$ and $Q_1, Q_2, \cdots$ be sequences of positive numbers such that $q_n \to 0$ and $Q_n \to \infty$ as $n \to \infty$. Choose positive integers n_1 and k_1 such that

$$| b(n_1, k_1) r_1(k_1) | > Q_1. \tag{15}$$

Choose an integer j_2 so great that $j_2 > k_1$ and

$$\sum_{k=j(2)}^{\infty} | b(n_1, k) r_1(k) | < q_1; \tag{16}$$

this is possible since $r_1 > R$ and the series in (13) is, for each n, absolutely convergent when $r > R$. Then choose n_2 and k_2 such that $k_2 > j_2$ and

$$| b(n_2, k_2) r_1(k_2) | > \sum_{k=0}^{k(1)} | b(n_2, k) r_1(k) | + Q_2. \tag{17}$$

When $n_1, k_1, n_2, k_2, \cdots, n_{h-1}, k_{h-1}$ have been determined, choose $j_h > k_{h-1}$ so great that

$$\sum_{k=j(h)}^{\infty} | b(n_{h-1,k}) r_1(k) | < q_{h-1}. \tag{18}$$

Then choose n_r and k_r such that $k_r > j_r$, $n_r > n_{r-1}$, and

$$|b(n_h, k_h)r_1(k_h)| > \sum_{k=0}^{k(h-1)} |b(n_h, k)r_1(k)| + Q_h. \tag{19}$$

We thus define, by induction, increasing sequences $n_1, n_2, \cdots$ and $k_1, k_2, \cdots$ of integers such that (19) and, by virtue of (18),

$$\sum_{k=k(h+1)}^{\infty} |b(n_h, k)r_1(k)| < q_h \tag{20}$$

hold for each $h=1, 2, \cdots$. Let $\sum u_k$ be the special series for which $u_k=r_1(k_h)$ when $k=k_h$, $h=1, 2, 3, \cdots$ and $u_k=0$ when $k\neq k_1, k_2, \cdots$. Then $\sum u_k z^k$ has radius of convergence $r_1>R$ and hence must, by our hypothesis, be summable B. But the transform σ_n of this series is such that, for each index h,

$$\begin{aligned} |\sigma(n_h)| &= \left| \sum_{k=0}^{\infty} b(n_h, k)u_k \right| \\ &= \left| b(n_h, k_h)r_1(k_h) + \sum_{k=0}^{k(h-1)} b(n_h, k)u_k + \sum_{k=k(h+1)}^{\infty} b(n_h, k)u_k \right| \qquad (21) \\ &\geqq |b(n_h, k_h)r_1(k_h)| - \sum_{k=0}^{k(h-1)} |b(n_h, k)r_1(k)| - \sum_{k=k(h+1)}^{\infty} |b(n_h, k)r_1(k)| \end{aligned}$$

and hence, because of (19) and (20), such that

$$|\sigma(n_h)| \geqq Q_h - q_h. \tag{22}$$

Therefore $|\sigma(n_h)| \to \infty$ as $h\to\infty$, and this contradiction of the conclusion that σ_n must be convergent completes the proof of the necessity of the conditions in Theorem 3.

In case the sequence $k_1, k_2, \cdots$ becomes infinite sufficiently rapidly, the series $\sum u_n$ which we constructed to prove necessity of (11) generates a gap power series $\sum u_n z^n$ whose circle of convergence is its natural boundary. It would be interesting to have criteria that $B\{\sum u_n\}$ exists whenever $\sum u_n z^n$ generates an analytic function with singularities belonging to smaller prescribed sets.

6. Proof of sufficiency of the conditions in Theorem 3. We now assume that (10) and (11) hold, and that $\sum u_n$ is a series for which $\sum u_n z^n$ has radius of convergence r_1 greater than R. We shall show that $B\{\sum u_n\}$ exists by proving (12).

Choose r such that

$$R < r < r_1. \tag{23}$$

Then $\sum u_n r^n$ converges absolutely. Let $M(r)$ be the constant in (11). Using (11), we obtain

$$|b_{nk}u_k| \leqq |b_{nk}r^{-k}|\,|u_k r^k| \leqq M(r)\,|u_k r^k| \tag{24}$$

and hence, using (10), also

$$|\beta_k u_k| \leqq M(r)\,|u_k r^k|. \tag{25}$$

Hence $\sum b_{nk}u_k$ and $\sum \beta_k u_k$ converge absolutely. Hence the B transform σ_n of $\sum u_n$ exists. Let $\sigma=\sum \beta_k u_k$. To show that $\sigma_n \to \sigma$, let $\epsilon>0$ and choose an index q such that

$$M(r)\sum_{k=q}^{\infty} |u_k r^k| < \frac{1}{4}\epsilon. \tag{26}$$

Then choose N such that

$$\sum_{k=0}^{q-1} |b_{nk}-\beta_k||u_k| < \frac{1}{2}\epsilon \qquad n > N; \tag{27}$$

this choice of N is made possible by (10). Then

$$\begin{aligned} |\sigma_n - \sigma| &\leqq \sum_{k=0}^{q-1} |b_{nk}-\beta_k||u_k| + \sum_{k=q}^{\infty} |b_{nk}-\beta_k||u_k| \\ &\leqq \sum_{k=0}^{q-1} |b_{nk}-\beta_k||u_k| + \sum_{k=q}^{\infty} |b_{nk}||u_k| + \sum_{k=q}^{\infty} |\beta_k||u_k| \end{aligned} \tag{28}$$

and hence $|\sigma_n-\sigma| <\epsilon$ when $n>N$. This completes the proof of Theorem 3.

7. Conditions for $B\{\sum u_n\}=\sum u_n$. We shall show that the following theorem follows easily from Theorem 3.

THEOREM 4. *Let $R\geqq 1$. In order that the series-to-sequence transformation B be such that $B\{\sum u_n\}=\sum u_n$ whenever the power series $\sum u_n z^n$ has radius of convergence greater than R, it is necessary and sufficient that*

$$\lim_{n\to\infty} b_{nk} = 1 \qquad k = 0, 1, 2, \cdots \tag{29}$$

and that, corresponding to each number r for which $r>R$, there exist a constant $M(r)$ such that

$$|b_{nk}| \leqq M(r)r^k \qquad n, k = 0, 1, 2, \cdots. \tag{30}$$

If B has the property in question, then Theorem 3 implies that (10) and (11) must hold. Application of B to the series $0+\cdots+0+1+0+\cdots$ treated in the first paragraph of section 5 shows that (29) holds, and necessity is proved. Sufficiency of the conditions in Theorem 4 is proved by setting $\beta_0=\beta_1=\cdots=1$ in the latter part of the statement of Theorem 3.

If, in the statement of Theorem 4, the condition $R\geqq 1$ were replaced by the condition $R\geqq 0$ of Theorem 3, the result would be illusory when $0\leqq R<1$. In the first place, if $0\leqq R<1$, the hypothesis that $\sum u_n$ has radius of convergence greater than R would not imply convergence of $\sum u_n$, and accordingly there could be no B such that $\sum u_n$ is summable B to the number to which it converges. In the second place, if $0\leqq R<1$, the requirement that (30) hold for each r greater than R would be inconsistent with the requirement that (29) hold.

8. A theorem of Schur and Szász. Schur [1918] proposed the problem of proving sufficiency of the condition (32) in the following theorem, and Szász [1922] proved the theorem. Okada [1929] proved sufficiency by use of contour integration.

THEOREM 5. *In order that a sequence* $b_0, b_1, \cdots$ *be such that*

$$\lim_{n\to\infty} \frac{b_n u_0 + b_{n-1}u_1 + \cdots + b_0 u_n}{b_n} = \sum u_n \tag{31}$$

whenever $\sum u_n z^n$ *has radius of convergence greater than* 1, *it is necessary and sufficient that*

$$\lim_{n\to\infty} b_{n-1}/b_n = 1. \tag{32}$$

The condition (32) implies and is implied by the set of conditions

$$\lim_{n\to\infty} b_{n-k}/b_n = 1 \qquad k = 0, 1, 2, \cdots. \tag{33}$$

To show that Theorem 5 is a corollary of Theorem 4, let

$$b_{nk} = b_{n-k}/b_n \qquad 0 \leqq k \leqq n \tag{34}$$

and $b_{nk}=0$ when $k>n$. By Theorem 4, case $R=1$, (31) holds for the stated class of series if and only if (33) holds and there corresponds to each θ in the interval $0<\theta<1$ a number $M(\theta)$ such that

$$\left| (b_{n-k}/b_n)\theta^k \right| \leqq M(\theta) \qquad 0 \leqq k \leqq n. \tag{35}$$

However, as we shall show, (32) implies (35); hence (32) alone serves as a necessary and sufficient condition. Let $b_{n-1}/b_n \to 1$, let $0<\theta<1$, and let $x_n=(b_{n-1}/b_n)\theta$. Then $x_n\to\theta$. Hence there is a number $A(\theta)$ such that $|x_n|<A(\theta)$ for all n. Moreover there is at most a finite set of values of n for which $|x_n|>1$; let $B(\theta)$ denote the number of such n's. Then, when $0\leqq k\leqq n$,

$$\left| (b_{n-k}/b_n)\theta^k \right| = \left| x_{n-k+1}x_{n-k+2}\cdots x_n \right| \leqq [A(\theta)]^{B(\theta)} \tag{36}$$

and (35) holds when $M(\theta)$ is the last member of (36). Thus (32) implies (35) and Theorem 5 is proved.

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AN INTERPOLATION FORMULA

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1. Introduction and preliminary material. In almost any discussion of interpolation formulas a certain collection of formulas are derived which express the interpolated value of a function in terms of certain of its values and its differences. These standard formulas are all expressions for the polynomial which coincides with the given function at certain values of the independent variable, and therefore must be transformable into each other. The purpose of this note is to show that they are all in fact instances of one and the same formula; that is, it is possible to write and prove a formula involving a number of parameters which, for various choices of the parameters, reduces to various standard formulas. Since the proof of this general formula is quite simple, one thus obtains the standard set of interpolation formulas (for functions tabulated at equal intervals, in terms of differences) without much effort.

In order to avoid questions of convergence, we shall suppose that a finite number of values of a function $F(z)$ are given corresponding to equally spaced values of the argument z, say $z=a, a+h, \cdots, a+mh$. Let $f(z)$ be the polynomial which coincides with $F(z)$ at these tabulated values. Our problem is to find convenient expressions for the polynomial $f(z)$ in terms of the tabulated values and their differences. We shall not discuss how well $f(z)$ represents the original function $F(z)$.

If we make the transformation $z=a+xh$, the polynomial $f(z)$ transforms into $g(x)=f(a+xh)$, and the tabulated values correspond to consecutive integral values of x. Thus our problem takes the form: Given the values, for $m+1$ consecutive integers, of a polynomial of degree m, to find convenient expressions for $g(x)$ in terms of the tabulated values and their differences. This form of the problem is notationally more convenient than the previous form, and is no less general.

We shall use the notation of forward differences. We define $\Delta^{(0)}g(x)$ to be $g(x)$ for each integer x, $\Delta^{(1)}g(x)=g(x+1)-g(x)$, and by induction

$$\Delta^{(k)}g(x) = \Delta^{(k-1)}g(x+1) - \Delta^{(k-1)}g(x). \tag{1}$$

A set of polynomials useful to us is defined by the equations

$$P_k(x) = x(x-1)(x-2)\cdots(x-k+1)/k!, \qquad k = 1, 2, \cdots,$$

(2) $$P_0(x) = 1,$$

$$P_k(x) = 0, \qquad k = -1, -2, \cdots.$$

These evidently satisfy

(3) $$P_k(x) = 0 \qquad (k > 0,\ x = 0, \cdots, k-1).$$

Another useful property of these polynomials, which will now be established, is

(4) $$\Delta^{(1)}P_k(x) = P_{k-1}(x).$$

If $k>0$, by (1) and (2)

$$\begin{aligned}\Delta^{(1)}P_k(x) &= P_k(x+1) - P_k(x)\\ &= [(x+1)(x)\cdots(x-k+2) - x(x-1)\cdots(x-k+1)]/k!\\ &= kx(x-1)\cdots(x-k+2)/k!\\ &= P_{k-1}(x),\end{aligned}$$

proving (4) for $k>0$. If $k\leqq 0$, $P_k(x)$ is constant, so its first difference is zero, and $k-1$ is negative, so $P_{k-1}(x)$ is also zero. Thus (4) holds for all integers k.

By repeated applications of (4), we find that for all nonnegative integers j the equation

(5) $$\Delta^{(j)}P_k(x) = P_{k-j}(x) \qquad (k = 0, \pm 1, \pm 2, \cdots)$$

is satisfied.

2. A lemma equivalent to Sheppard's Rule. It is now easy to establish the following lemma.

LEMMA. *Let $c_0, c_1, c_2, \cdots$ be a sequence of integers such that each difference $c_{h+1}-c_h$ $(h=1, 2, \cdots)$ is either 0 or 1. Let $g(x)$ be a polynomial, and m its degree. Then for all x*

(6) $$g(x) = \sum_{j=0}^{m} P_j(x + c_j)\Delta^{(j)}g(-c_{j+1}).$$

Let $g(x)$ be written in the form

(7) $$g(x) = a_0 + a_1P_1(x + c_1) + \cdots + a_mP_m(x + c_m).$$

This is evidently possible; the coefficient of x^m on the right is $a_m/m!$, and this can be made equal to the coefficient of x^m in the left member by proper choice of a_m. Transposing the term $a_mP_m(x+c_m)$ to the left and repeating the argument gives the value of a_{m-1}, and so on. Taking kth differences of both members of (7) yields, with (5),

(8) $$\Delta^{(k)}g(x) = \sum_{j=0}^{m} a_jP_{j-k}(x + c_j).$$

In the sum on the right, all terms with $j<k$ vanish by (2). The term with $j=k$ is a_k, also by (2). In (8) we now set $x=-c_{k+1}$. By the choice of the integers c_k, each term of the sum

$$(c_j - c_{j-1}) + (c_{j-1} - c_{j-2}) + \cdots + (c_{k+2} - c_{k+1}) \qquad (j > k+1)$$

is either zero or 1, so that

$$0 \leqq c_j - c_{k+1} \leqq j - k - 1 \qquad (j > k).$$

But this, together with (3), implies that

$$P_{j-k}(-c_{k+1} + c_j) = 0 \qquad (j > k).$$

Thus all that remains of (8) is

$$\Delta^{(k)}g(-c_{k+1}) = a_k,$$

and (7) takes the form (6), establishing the lemma.

Equation (6) can be stated as a simple rule for writing a class of interpolation formulas. Write down the successive differences of $g(x)$, the difference of order 0 (that is, g itself) being chosen on any line and each succeeding difference being either on the same line as its predecessor or else on the line above. Multiply each $\Delta^{(k)}g$ by a coefficient whose denominator is $k!$ and whose numerator is the product of k factors, the first being $x-$(argument of $\Delta^{(k-1)}g$) and the others decreasing by steps of 1. This is essentially the same as Sheppard's rule.* Among others, it yields the Newton-Gregory "forward" and "backward" formulas and the Newton-Gauss "forward" and "backward" formulas, as is known and will again be shown in the collection of special cases following the next theorem.

3. A more general formula. Although the lemma yields several interesting special cases, there are others which it does not cover, so we now proceed to establish a more general formulation.

THEOREM. *Let $c_{-1}, c_0, c_1, \cdots$ be a sequence of integers such that each difference $c_{k+1}-c_k$ is either 0 or 1. Let $g(x)$ be a polynomial, and m its degree. Then the equation*

$$(9) \qquad g(x) = \sum_{j=0}^{m} [P_j(x+c_j) + y_j P_{j-1}(x+c_{j-1})][\Delta^{(j)}g(-c_{j+1}) - y_{j+2}\Delta^{(j+1)}g(-c_{j+2})]$$

is valid for all numbers $x, y_0, y_1, \cdots, y_{m+1}$ such that no two consecutive y_j are both different from zero.

Upon expansion of its right member, equation (9) becomes

---

* H. Freeman, An Elementary Treatise on Actuarial Mathematics, page 73.

$$g(x) = \sum_{j=0}^{m} P_j(x + c_j)\Delta^{(j)}g(-c_{j+1})$$
$$+ y_0 P_{-1}(x + c_{-1})g(-c_1) - y_{m+1}P_m(x + c_m)\Delta^{(m+1)}g(-c_{m+2})$$
$$+ \sum_{j=1}^{m} y_j[-P_{j-1}(x + c_{j-1})\Delta^{(j)}g(-c_{j+1}) + P_{j-1}(x + c_{j-1})\Delta^{(j)}g(-c_{j+1})]$$
$$- \sum_{j=0}^{m} y_j y_{j+1} P_{j-1}(x + c_{j-1})\Delta^{(j+1)}g(-c_{j+2}).$$

But $P_{-1}(x+c_{-1})$ is zero by definition (2), $\Delta^{(m+1)}g(x)=0$ because $g(x)$ is of degree m, and $y_jy_{j+1}=0$ by the hypotheses of the theorem. Also, each term of the second summation in the right member of the expanded form of equation (9) vanishes identically. Hence the y_j all drop out of formula (9), and the theorem is seen to be equivalent to the lemma, which has already been established.*

4. Applications. We now proceed to write some special cases of equation (9). If we choose all $c_h=0$, then by the lemma or the theorem with all $y_j=0$ we find

$$g(x) = g(0) + x\Delta^{(1)}g(0) + \cdots + [x(x-1)\cdots(x-k+1)/k!]\Delta^{(k)}g(0) + \cdots, \tag{10}$$

which is the Newton-Gregory formula, or more specifically the Newton-Gregory "forward" formula. If we choose $c_h=h-1$, by the lemma or by the theorem with all $y_j=0$

$$g(x) = g(0) + x\Delta^{(1)}g(-1) + \cdots + [(x+k-1)\cdots(x)/k!]\Delta^{(k)}g(-k) + \cdots, \tag{11}$$

which is the Newton-Gregory "backward" formula.

If we choose $c_{2k}=c_{2k+1}=k$, equation (6), or equation (9) with all y_j set equal to 0, yields

$$g(x) = g(0) + x\Delta^{(1)}g(-1) + [(x+1)x/2!]\Delta^{(2)}g(-1) + \cdots + [(x+k)\cdots(x-k+1)/(2k)!]\Delta^{(2k)}g(-k) + [(x+k)\cdots(x-k)/(2k+1)!]\Delta^{(2k+1)}g(-k-1) + \cdots, \tag{12}$$

which is the Newton-Gauss "backward" formula.

The remaining five formulas which we shall discuss all are based on the choice

$$c_{2k-1} = c_{2k} = k - 1. \tag{13}$$

Substitution in (6) yields

* This arrangement of the proof was suggested by the referee.

$$
\begin{aligned}
g(x) &= g(0) + x\Delta^{(1)}g(0) + [x(x-1)/2!]\Delta^{(2)}g(-1) + \cdots \\
&\quad + [(x+k-1)\cdots(x-k)/(2k)!]\Delta^{(2k)}g(-k) \\
&\quad + [(x+k)\cdots(x-k)/(2k+1)!]\Delta^{(2k+1)}g(-k) + \cdots,
\end{aligned} \tag{14}
$$

which is the Newton-Gauss "forward" formula.

In order to apply (9) with non-zero values of y_j, it is convenient first to perform a certain amount of algebraic manipulation. The first of the following four formulas holds for $k>0$, the others for $k\geqq 0$.

$$
\begin{aligned}
&P_{2k}(x+k-1) + y_{2k}P_{2k-1}(x+k-1) \\
&\qquad = (x+k-1)\cdots(x-k+1)[x-k+2ky_{2k}]/(2k)!,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&P_{2k+1}(x+k) + y_{2k+1}P_{2k}(x+k-1) \\
&\qquad = (x+k-1)\cdots(x-k)[x+k+(2k+1)y_{2k+1}]/(2k+1)!,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\Delta^{(2k)}g(-k) - y_{2k+1}\Delta^{(2k+1)}g(-k) \\
&\qquad = -y_{2k+1}\Delta^{(2k)}g(-k+1) + (1+y_{2k+1})\Delta^{(2k)}g(-k),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&\Delta^{(2k+1)}g(-k) - y_{2k+2}\Delta^{(2k+2)}g(-k-1) \\
&\qquad = (1-y_{2k+k})\Delta^{(2k+1)}g(-k) + y_{2k+2}\Delta^{(2k+1)}g(-k-1).
\end{aligned} \tag{18}
$$

The first two of these follow at once from (2); the second two are immediate consequences of the definition (1).

The c_k being as in (13), let us select $y_{2k}=1/2$, $y_{2k+1}=0$. With the help of (15) and (18), (9) reduces to

$$
\begin{aligned}
g(x) &= g(0) + x[\Delta^{(1)}g(0) + \Delta^{(1)}g(-1)]/2 + \cdots \\
&\quad + \frac{(x+k-1)\cdots(x-k+1)[x]}{(2k)!}\Delta^{(2k)}g(-k) \\
&\quad + P_{2k+1}(x+k)[\Delta^{(2k+1)}g(-k) + \Delta^{(2k+1)}g(-k-1)]/2 \\
&\quad + \cdots \\
&= g(0) + \cdots + \frac{x^2(x^2-1^2)\cdots(x^2-\overline{k-1}^2)}{(2k)!}\Delta^{(2k)}g(-k) \\
&\quad + P_{2k+1}(x+k)[\Delta^{(2k+1)}g(-k) + \Delta^{(2k+1)}g(-k-1)]/2 \\
&\quad + \cdots,
\end{aligned} \tag{19}
$$

which is Stirling's formula.

If we select $y_{2k}=0$, $y_{2k+1}=-1/2$, with the help of (16) and (17) equation (9) reduces to

$$
\begin{aligned}
g(x) &= [g(0)+g(1)]/2 + [x-1/2]\Delta^{(1)}g(0) + \cdots \\
&\quad + P_{2k}(x+k-1)[\Delta^{(2k)}g(-k) + \Delta^{(2k)}g(-k+1)]/2 \\
&\quad + \frac{[x-1/2](x+k-1)\cdots(x-k)}{(2k+1)!}\Delta^{(2k+1)}g(-k) + \cdots,
\end{aligned} \tag{20}
$$

which is Bessel's formula.

If we select $y_{2k}=0$, $y_{2k+1}=-(x+k)/(2k+1)$, the right member of (16) vanishes, so the terms in (9) corresponding to odd j all disappear, and (9) takes the form

$$
\begin{aligned}
g(x) = {} & xg(1) + (1-x)g(0) + \cdots \\
& + P_{2k}(x+k-1)\left[\frac{(x+k)}{2k+1}\Delta^{(2k)}g(-k+1)\right. \\
& \left. + \left\{1 - \frac{x+k}{2k+1}\right\}\Delta^{(2k)}g(-k)\right] \\
& + \cdots .
\end{aligned} \tag{21}
$$

This expansion can be given a more symmetric form if we first establish the identity

$$
P_n(x+m) = (-1)^n P_n(n-m-1-x). \tag{22}
$$

The truth of this statement is easily verified, since the factors on the left are the negatives of those on the right taken in the reversed order. Using this,

$$
\begin{aligned}
P_{2k}(x+k-1)\left\{1 - \frac{x+k}{2k+1}\right\} &= -P_{2k+1}(x+k-1) \\
&= +P_{2k+1}(k+1-x).
\end{aligned}
$$

It is customary to introduce a symbol, say ξ, for $1-x$. If this is done, equation (21) becomes

$$
\begin{aligned}
g(x) = {} & xg(1) + \xi g(0) + \cdots \\
& + P_{2k+1}(x+k)\Delta^{(2k)}g(-k+1) + P_{2k+1}(\xi+k)\Delta^{(2k)}g(-k) \\
& + \cdots .
\end{aligned} \tag{23}
$$

This is Everett's formula.

Finally we select $y_{2k+1}=0$, $y_0=0$, $y_{2k}=(k-x)/2k$. By (15), all terms in (9) corresponding to even values of $j>0$ disappear, and (9) takes the form

$$
\begin{aligned}
g(x) = g(0) + \sum_{k=0} P_{2k+1}(x+k)\left[\left\{1 - \frac{k+1-x}{2k+2}\right\}\Delta^{(2k+1)}g(-k)\right. \\
\left. + \frac{k+1-x}{2k+2}\Delta^{(2k+1)}g(-k-1)\right].
\end{aligned} \tag{24}
$$

With (22), this becomes

$$
\begin{aligned}
g(x) = g(0) + \sum_{k=0} [P_{2k+2}(x+k+1)\Delta^{(2k+1)}g(-k) \\
- P_{2k+2}(k+1-x)\Delta^{(2k+1)}g(-k-1)].
\end{aligned} \tag{25}
$$

This is Steffensen's formula.

PRIME-REPRESENTING FUNCTIONS

R. C. BUCK, Harvard University

If $f(x)$ has the property that $f(n)$ is a prime number for $n=1, 2, \cdots$, then $f(x)$ is called a prime-representing function. If further we require that there be an infinite number of *distinct* primes among the values $f(n)$, then we may call $f(x)$ a *proper* prime-representing function. There are analytic functions having this property. However, there is reason to believe that no simple function, finitely expressible, has this property. Fermat conjectured falsely that $f(x)=2^{2^x}+1$ was such a function. It is well known that no polynomial can represent primes, unless it is constant and therefore improper. The object of this note is to offer a simple proof that no rational function can represent primes.

LEMMA. *If $P(x)$ is a polynomial, and $P(n)=A_n+\gamma_n$ where A_n is integral and* $\lim \gamma_n=0$, *then for large n, all the γ_n vanish.*

Let $P(x)$ be of degree $m-1$, so that $\Delta^m P(x)\equiv 0$. Putting $x=n$, we have $\Delta^m\gamma_n=-\Delta^m A_n$, for each n. Since A_n is integral, so is $\Delta^m A_n$ and therefore $\Delta^m\gamma_n$. Writing this as $\Delta^m\gamma_n=\sum_0^m\binom{m}{k}(-1)^{m-k}\gamma_{n+k}$, and using $\lim \gamma_n=0$, we have $\lim_{n\to\infty}\Delta^m\gamma_n=0$. But, $\Delta^m\gamma_n$ is integral and consequently for some n_0, we have $\Delta^m\gamma_n=0$ for all $n>n_0$. Now set

$$Q(x)=\sum_{k=0}^{\infty}\frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}\Delta^k\gamma_{n_0}.$$

This is a polynomial of degree m at most, for if $k\geqq 1$,

$$\begin{aligned}\Delta^{m+k}\gamma_{n_0} &= \Delta^k\Delta^m\gamma_{n_0}\\ &= \sum_{\lambda=0}^{k}\binom{k}{\lambda}(-1)^{k-\lambda}\Delta^m\gamma_{n_0+k}.\\ &= 0.\end{aligned}$$

Then, $Q(n)=\gamma_{n+n_0}$ and since $\lim \gamma_n=0$, we must have $Q(x)\equiv 0$; thus, we have proved that $\gamma_n=0$ for $n>n_0$.

THEOREM. *If a rational function takes prime values at the integers, it must be constant.*

Let $R(x)=P(x)+g(x)/h(x)$ with $P(x)$, $g(x)$, $h(x)$ polynomials, and the degree of $g(x)$ less than that of $h(x)$. Let $R(n)=A_n$, a prime, for each $n\geqq 1$. If we set γ_n equal to $g(n)/h(n)$, then $\lim \gamma_n=0$. Then, we have $P(n)=A_n-\gamma_n$. Applying the lemma, we have $\gamma_n=0$ for large n, and thus $P(n)=A_n$, a prime, for all $n>n_0$. But, as we have noted before, this is impossible unless $P(x)$ is a constant. Finally, since $g(n)/h(n)=0$ for $n>n_0$, we see that $R(x)$ reduces to a constant.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

The River Mathematics. By A. Hooper. Henry Holt and Co., 1945. 8+401 pages. $3.75.

"The object of this book," writes the author in his opening line, "is to trace in a simple way the development of mathematical ideas and processes from their insignificant source to the mighty flood on which now sail the proud ships of Science, Engineering, and Aeronautics, ..." And again, a few lines later, "Both the casual visitor and more serious newcomer will find in this book a bird's-eye view of mathematical ideas."

Unfortunately the reader, whether casual or serious, will find in this book little more than a revised and enlarged version of the author's earlier and popular *Mathematics Refresher.* And the material of that book, successful though it may have been in the urgent task of providing many young men with a stock of elementary manipulative skills, is hardly the stuff to use in tracing the development of mathematical ideas.

Many of the chapters start in a relatively promising way with a discussion of the derivation of terms and with similar matters of general historical interest. But most of them, after too few pages of exposition of this sort, revert to collections of miscellaneous facts, rules, and illustrative examples, much of which material—even to certain identical methods of presentation and identical examples—is to be found in the *Refresher.* (It may be added that the educational effectiveness of the earlier work is lost, in the present volume, through the omission of exercises and self-tests.)

The reason for the inclusion of some of the new material is not clear. Chapter 14, for example, is an abbreviated conventional presentation of the analytic geometry of the straight line and the conic sections. It contributes no more to the development of mathematical ideas than does any ordinary text which includes the obvious applications of the conics to searchlights, telescopes, bridges, and such. In the same way, the all too brief and again conventional treatment of the calculus in Chapter 15 does little more than any ordinary text to increase the general reader's understanding of the development, nature, and significance of that branch of mathematics.

Many exciting promises are made in the opening chapter, which discusses in attractive detail the objectives of the book. But the remaining chapters leave these promises largely unfulfilled. Indeed, there are several good books, already on the market, which come closer than this book comes to meeting the stated objectives.

E. P. NORTHROP

An Introduction to Mathematics for Teachers. By L. E. Boyer. New York, Henry Holt and Co., Inc., 1945. 17+478 pages. $3.25.

One of the commonest problems of those who prepare teachers of elementary and secondary school mathematics is that the teacher candidates are found to have a very limited understanding of the subjects they hope to teach. Students may exhibit considerable skill in algebraic manipulation, formal differentiation, and integration, and yet be pitifully unprepared to explain "what mathematics is all about" to pupils in the lower schools. The author of this book has sought to help in solving this problem by providing under one cover several different sorts of material which should be mastered by teachers. In the preface he says: "The purpose of this book is to provide prospective elementary and secondary school teachers . . . some means of learning about the nature, significance and use of mathematics from early times to the present." Although apparently intended for use in colleges, it presupposes only the background possessed by students who "ended their formal study of mathematics in grade nine . . . ."

This is not a textbook for courses on methods of teaching. The content includes topics from arithmetic, elementary algebra, geometry, and trigonometry. The book exemplifies the tendency of recent books on general mathematics at the college level to provide generous expository discussion. Both the historical setting and practical applications of many of the topics are developed in the text. The exercises for students are numerous and varied for this sort of book, and the practical type of problem is more plentiful than usual. Although it is not entirely free from questionable statements and minor blemishes of style, none will seriously interfere with the usefulness of the book. On the whole, the author has done an interesting and effective piece of writing for the type of student he has in mind.

The popularity of books of this kind seems to be growing rapidly. In the case of this text, nearly all of the content is definitely of the secondary school level, and ideally prospective teachers (and perhaps college students in general) should be expected to be quite familiar with the ideas before they *enter* college. Normally the notions would be developed slowly and study would be extended over several years of high school. To achieve so many objectives in a single course in college is a formidable task. At present, far too many students have little understanding of some of these topics when they are graduated from college and are permitted to teach. When this situation has ultimately been remedied, in part by courses based on books like this one, there will be a better chance for the high schools to do this job properly. The colleges may then devote their efforts to a broader range of content and to a treatment of many of the same ideas suitable for students of greater mathematical maturity.

M. L. HARTUNG

Curso de Análisis Matemático. Tomo I. Por Cristóbal de Losada y Puga. Lima, Universidad Católica del Perú, 1945. 26+632 pp. $8.00.

This is the first volume of a treatise which the author intends to be comparable with the classical French *Traité d'Analyse.* At the same time, he has attempted to organize it in such a way that selected portions can be used for a first course in calculus. To this end he has adopted an approach in which a number of topics are treated from an elementary and from an advanced point of view in consecutive sections. The North American reader will be particularly interested in the work as a source of alternative approaches to the teaching of familiar topics. It is salutary to realize that neither the arrangement of material, the method of proof, nor the illustrative examples have to be those to which one has become accustomed.

This volume is divided into four parts: Introduction, Differentiation, Integration, Introduction to Differential Equations. The first part begins with a rapid introduction to the number concept and point sets. Chapters on variables and functions and on limits follow; these are elementary in character, though going beyond most North American introductions to the calculus in proving, for example, that the limit usually taken as the definition of e actually exists. A supplementary chapter discusses continuous functions from the advanced point of view, with careful proofs.

The second part discusses differentiation of the elementary functions, properties of derivatives, and infinitesimals, in the same fashion. (It is unfortunate that the author gives Duhamel's theorem in the traditional fallacious form.) The usual applications of differentiation (tangents, maxima and minima, velocities) follow; the exposition is not particularly detailed. Functions of several variables and partial derivatives are studied next, with a careful discussion of continuity and a proof of the implicit function theorem.

The third part opens with two chapters on integration, containing the topics customary in a first course in calculus. The elegant geometric interpretation of integration by parts was new to the reviewer. A supplementary chapter proves an existence theorem for the Riemann integral, introduces the Stieltjes integral, and gives a brief account of Lebesgue integration. It seems unfortunate that the author does not give more explanation of why these more general integrals are useful; it is to be hoped that he will do so in a later volume in connection with the study of series. A chapter on systematic integration culminates in a detailed presentation of elliptic integrals and their transformation. A chapter on applications of integration contains, besides the customary material, a discussion of the length of a rectifiable curve as the limit of inscribed polygons.

The fourth part carries the elementary theory of differential equations as far as the solution of the inhomogeneous linear equation with constant coefficients. Emphasis is placed on the geometric interpretation of a differential equation and on physical applications (outside of dynamics: the author believes that dynamical applications are better left until the student has begun a systematic study of mechanics).

In this first volume, the author has not gone into as great detail as the treatises which he has taken as his inspiration, but his exposition is more leisurely. In general, he has succeeded admirably in his aim of combining clarity with rigor. There are numerous historical remarks, and extended discussion of such interesting historical topics as Archimedes' quadrature of the parabola and Wren's rectification of the cycloid. The book is well printed, and the numerous diagrams deserve special mention for their high quality.

R. P. BOAS, JR.

Higher Algebra for Schools. By W. L. Ferrar. Oxford University Press, 1945. 7+214 pages. $3.75.

The subject matter of this book corresponds roughly to that treated by numerous American textbooks bearing the title College Algebra. However, the book differs from our college algebras in that it presents fewer topics but develops them at greater length.

Fully one-third of the book is devoted to the theory of polynomials and equations. There are two chapters on graphs, one on mathematical induction, and two chapters on infinite series with emphasis on the binomial series and the exponential and logarithmic functions. The concluding two chapters on linear equations and determinants do not go beyond determinants of order 4, but include such advanced topics as the rule for differentiating a determinant and the rule for multiplying two determinants.

It seems odd to an American that an algebra book which presupposes a knowledge of differentiation and integration of easy functions should omit Demoivre's theorem and allied topics. Again, the theory of permutations and combinations is treated briefly, and then primarily for the purpose of presenting the combinatorial proof of the binomial theorem; and there is no mention of probability. One also misses Descartes' rule and Horner's method, although Newton's method is included. These topics will presumably be treated in a subsequent volume promised by the author.

The book contains an abundance of illustrative examples and exercises involving polynomial identities, undetermined coefficients, relations among binomial coefficients, factorization of determinants and many formal devices. An ambitious student will find these problems a valuable aid in cultivating algebraic techniques. Some of the exercises are labeled "hard" and others "harder." In the reviewer's opinion many of the exercises not so labeled would drive our average freshman to despair. But this remark is no reflection on the author, who has written an excellent book for a selected group of British students whose mathematical preparation and traditions are different from those of our freshmen.

LOUIS WEISNER

NEW BOOKS RECEIVED

Analytic Geometry. Third Edition. By F. S. Nowlan. New York, McGraw-Hill Book Co., 1946. 14+355 pages. $2.75.

Elements of Calculus. By W. A. Granville, P. F. Smith, and W. R. Longley. Boston, Ginn and Co., 1946. 11+549 pages. $3.75.

The Parallel Postulate of Euclid. By G. H. Draper. Arlington, Draper, 1946. 8 pages. $0.25.

A Collection of Papers in Memory of Sir William Rowan Hamilton. Scripta Mathematica Studies, No. 2. New York, Scripta Mathematica, 1945. 82 pages. $1.00.

International Payments; A Science. By R. O. Hall. Washington, D. C., Storage Bookshop, 1946. 11+124 pages.

Mathematical Cuneiform Texts. (American Oriental Series, vol. 29) New Haven, American Oriental Society and American Schools of Oriental Research, 1945. 10+173+49 pages. $5.00.

Mathematics for Technical and Vocational Schools. Third Edition. By Samuel Slade and Louis Margolis. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1946. 7+532 pages. $2.50.

Wiley Trigonometric Tables. Second Edition. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1945. 5+117 pages. $1.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 721. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Given two equal regular dodecagons. Show how to dissect one of them into twelve congruent pieces which can be fitted to the other to form another larger regular dodecagon.

E 722. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a square of four digits such that if one interchanges the two middle digits the new number is a square for radix 9.

E 723. *Proposed by A. W. Goodman, Republic Aviation Corporation*

Let A and B be any real quantities, and let k be a positive integer. Show that

$$\left|\frac{\cos kB \cos A - \cos kA \cos B}{\cos B - \cos A}\right| \leqq k^2 - 1, \tag{1}$$

$$\left|\frac{\cos kB - \cos kA}{\cos B - \cos A}\right| \leqq k^2, \tag{2}$$

equality holding only in the limit as B and A approach zero.

E 724. *Proposed by N. J. Fine, University of Pennsylvania, and Ivan Niven, Purdue University*

Define an n-admissible number k as one such that an n-dimensional cube may be subdivided into k cubes. Prove that for each n there exists an integer A_n such that all integers exceeding A_n are n-admissible.

E 725. *Proposed by Henry Scheffé, Princeton, N. J.*

If two events have probabilities p_1 and p_2 $(0<p_i<1)$, and correlation coefficient ρ, show that the range of possible ρ is the interval $-f(a_1a_2) \leqq \rho \leqq f(a_1/a_2)$, where $a_i=[p_i/(1-p_i)]^{1/2}$ and $f(x)=\min(x, 1/x)$. The correlation coefficient between two events E_1 and E_2 may be defined as that between x_1 and x_2, where $x_i=1$ if E_i happens and $x_i=0$ if not -E_i happens.

SOLUTIONS

Iterated Factorials

E 684 [1945, 395]. *Proposed by Paul Erdös, Stanford University*

Prove that $n!^{(n-1)!}$ divides $(n!)!$; and that if n is not a power of a prime, $n!^{n(n-2)!}$ divides $(n!)!$.

Solution by Leo Moser, University of Toronto. The number of times that a prime p divides $m!$ is given by $\sum_{i=1}^{\infty}[m/p^i]$.

For the first part all we need show is that $\sum[n!/p^i] \geqq (n-1)!\sum[n/p^i]$, which is clearly true.

In the second part, since n is not a power of a prime, the number of times that p divides $n!$ is the same as the number of times that p divides $(n-1)!$. Hence $\sum[n/p^i]=\sum[(n-1)/p^i]$. We therefore have

$$\sum[n!/p^i] \geqq n(n-2)!\sum[(n-1)/p^i] = n(n-2)!\sum[n/p^i],$$

which proves the theorem.

The first part may be generalized as follows: *If* $ab \leqq n!$, *then* $a!^b$ *divides* $(n!)!$. The second part may be generalized in this way: *If none of* $n, n-1, n-2, \cdots, n-k+1$ *is a power of a prime, then* $n!^{n!/(n-k)}$ *divides* $(n!)!$. The proofs are similar to those given above.

Also solved by the proposer.

The Careless Professor

E 686 [1945, 457]. *Proposed by E. D. Schell, Arlington, Virginia*

The professor writes out a series of positive terms, $\sum a_n$, on the blackboard. Because of his carelessly written addition signs, his students consider instead the product $\prod a_n$. Unfortunately the misunderstanding cannot be discovered even though he calls out the partial sum as he writes each term. Given that a_1 and a_2 are integers, find the first five terms.

Solution by Roy Dubisch, Montana State University. Since a_1 and a_2 are integers, the relation $a_1+a_2=a_1a_2$ implies that $a_1=a_2=2$. Then $a_1+a_2+a_3=a_1a_2a_3$ implies that $a_3=4/3$. Similarly, we obtain $a_4=16/13$ and $a_5=256/217$.

While it is easy to show that the numerator of a_n $(n>1)$ is given by 2^t, $t=2^{n-2}$, it appears more difficult to construct a formula for the denominator of a_n. Even an iterative formula would seem to involve the partial sums (or products).

Also solved by D. W. Alling, Murray Barbour, W. G. Brady, R. E. Crane, J. H. Cross, Monte Dernham, M. P. de Regt, George Grossman, V. L. Klee, Jr., Margaret Olmstead, P. A. Piza, A. P. Rhodes, E. P. Starke, W. R. Talbot, F. Underwood, R. H. Urbano, Maud Willey, L. B. Zeropa, and the proposer.

Dernham and Underwood observed that for $n>1$, $\{a_n\}$ is a monotone decreasing sequence with limit 1.

A Fermagoric Triangle

E 688 [1945, 457]. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

Consider the right triangle ABC with sides $a=12$, $b=5$, $c=13$. $(a^2+b^2=c^2.)$ Keeping the base BC fixed, displace the vertex A without altering the perimeter (*i.e.*, along an ellipse with foci B and C) until the value of $\cos A$ is reduced from 5/13 to 7/38. Show that the sides of the new triangle satisfy the relation

$$a^3 = b^3 + c^3.$$

I. *Solution by F. L. Celauro, Lehigh University.* The perimeter of the new triangle is $a+b+c$, or 30. Since $a=12$, then $c=18-b$. Using the law of cosines, $\cos A=(b^2+c^2-a^2)/2bc$, with $\cos A=7/38$, we have

$$7/38 = [b^2 + (18 - b)^2 - 12^2]/2b(18 - b).$$

There are two solutions to this equation, $b=9\pm\sqrt{5}$. The corresponding values of c are $9\mp\sqrt{5}$. Since

$$12^3 = (9 \pm \sqrt{5})^3 + (9 \mp \sqrt{5})^3$$

the sides of the new triangle satisfy the required relation.

II. *Solution by Maud Willey, Denton, Texas.* In the new triangle $a=12$, $b+c=18$, $\cos A=7/38$, and (by the law of cosines)

$$a^2 = (b + c)^2 - 2bc(\cos A + 1).$$

Substituting the known values and solving for bc we find $bc=76$. Now

$$b^3 + c^3 = (b + c)^3 - 3bc(b + c)$$
$$= 18^3 - 3\cdot 76\cdot 18 = 12^3 = a^3.$$

Also solved by Frank Arena, LeRoy Babcock, Murray Barbour, James Carlisle, W. B. Clarke, M. P. de Regt, Archibald Henderson, Sam Kravitz, Albert Medwid, C. T. Oergel, Margaret Olmstead, A. P. Rhodes, J. T. Rodriguez, F. W. Saunders, E. D. Schell, E. S. Smith, Mary Sultzer, W. R. Talbot, J. A. Tierney, F. Underwood, R. H. Urbano, Jeanette Van Os, Hazel Wilson, B. F. Yanney, and the proposer.

Henderson observed that the new triangle is "Glenie's Triangle," as treated by J. Glenie in *The Antecedental Calculus* (London, 1793). As further references he gave L. E. Dickson's *History of the Theory of Numbers*, II, p. 546, and p. 33 of the proposer's recent book *Fermagoric Triangles*.

The proposer pointed out that in his book *Fermagoric Triangles* he establishes an identity equivalent to

$$(2rs)^3 = (r^2 + \sqrt{r(4s^3 - r^3)/3})^3 + (r^2 - \sqrt{r(4s^3 - r^3)/3})^3.$$

For $r=3$, $s=2$ we obtain the triangle considered in the problem.

The proposer also pointed out that if BC is kept fixed and the vertex A moved along the ellipse so that $\cos A$ is successively reduced from 5/13 to $(\sqrt{200340}-360)/(810-\sqrt{200340})$, $(\sqrt{62856}-234)/(324-\sqrt{62856})$, 7/38 we obtain three triangles in which, respectively, $a^5=b^5+c^5$, $a^4=b^4+c^4$, $a^3=b^3+c^3$. It is easily shown that these triangles are constructible by euclidean tools. We thus have here, along with the given right triangle, a set of isoperimetric "fermagoric" triangles, of orders 2, 3, 4, 5, having a common base BC, and all of them constructible by euclidean tools.

Four Collinear Points

E 689 [1945, 457]. *Proposed by Morgan Ward, California Institute of Technology*

Let A, B, C, D be four collinear points in the order written, and let P be any other point in space. Prove that the inequality

$$PA + PD \geqq PB + PC$$

holds for all positions of P if and only if $AB=CD$.

Solution by E. J. Stulken, Geophysical Service Inc., Dallas, Texas. Consider $AB=CD$. Designate the midpoint of AD by O and extend PO its own length to the point P_1. Then $PA=P_1D$, $PB=P_1C$, $PC=P_1B$, $PD=P_1A$. If P is on A, D, or on AD extended, $PA+P_1A=PB+P_1B$. If P is elsewhere, $PA+P_1A>PB+P_1B$. Therefore $PA+PD\geqq PB+PC$.

With $AB\neq CD$ and P beyond the longer segment on AD extended, $PA+PD<PB+PC$.

Also solved by Murray Barbour, J. B. Kelly, V. L. Klee, Jr., and the proposer.

An Extension of E 614

E 690 [1945, 457]. *Proposed by F. J. Duarte, Caracas, Venezuela*

If p is a prime number greater than 3, prove that the polynomial

$$(x + y)^p - x^p - y^p$$

is divisible by $x^3+2x^2y+2xy^2+y^3$. (*Cf.* E 614 [1944, 591].)

Solution by F. W. Saunders, student, University of North Carolina. It was proved by C. D. Olds (*cf.* E 614 [1944, 591]) that $(x+y)^p-x^p-y^p$ is divisible by x^2+xy+y^2 for $p\equiv 1$ or 5 (mod 6).

Now, for every prime $p>3$, we have $p\equiv 1$ or 5 (mod 6). Also

$$x^3 + 2x^2y + 2xy^2 + y^3 = (x + y)(x^2 + xy + y^2).$$

Moreover, $(x+y)^p-x^p-y^p$ is always divisible by $x+y$ for odd p. Therefore, since $x+y$ is prime to x^2+xy+y^2, the theorem follows.

Also solved by E. D. Schell, E. P. Starke, and the proposer. P. A. Piza gave the following reference from Vol. II of Dickson's *History of the Theory of Numbers:* "G. Heppel [Math. Quest. Educ. Times, 40, 1884, 124] proved that, if n is a prime>3, $(x+y)^n-x^n-y^n$ is divisible by $nxy(x+y)(x^2+xy+y^2)$ and found the coefficients of the general term of the quotient."

Arrow over Bridge

E 691 [1945, 516]. *Proposed by E. K. Paxton, Washington and Lee University*

The top of a certain bridge is 215 feet above the ground and 100 feet wide. Neglecting air resistance, find the minimum initial velocity with which an arrow is shot in order to go over the bridge. At what distance from the base must the archer stand? What is the angle of elevation? How high does the arrow go? (Assume the arrow leaves the bow at a height of 5 feet above the ground.)

I. *Solution by G. A. Williams, Oregon State College.* Assuming the plane of the trajectory to be perpendicular to the bridge and taking as axes the horizontal line in this plane 5 feet above the ground and the vertical line at the center of the bridge, the equation of the trajectory is

$$y = v_0^2 (\sin^2 \alpha)/2g - (x^2 g \sec^2 \alpha)/2v_0^2. \tag{1}$$

For $x=50$ we have $y=210$. Substituting these values in (1) and rearranging terms gives v_0 as a function of α:

$$v_0^4 \sin^2 \alpha - 420gv_0^2 - 2500g^2 \sec^2 \alpha = 0. \tag{2}$$

Differentiating with respect to α and setting $dv_0/d\alpha=0$, gives

$$2v_0^4 \sin \alpha \cos \alpha - 5000g^2 \sec^2 \alpha \tan \alpha = 0,$$

or

(3) $$v_0 = 5\sqrt{2g}\sec\alpha.$$

Substituting this in (2) and solving for $\tan\alpha$ gives

(4) $$\tan\alpha = \sqrt{47/5}, \quad \text{or} \quad \alpha = 71°56.1'.$$

Substituting from (4) in (3) we find

(5) $$v_0 = 2\sqrt{130g}.$$

Substituting (4), (5), and $x=0$ in (1) we obtain $y=235$ feet. Substituting (4), (5), and $y=0$ in (1) we obtain $x=10\sqrt{235}$. Hence the archer must stand $(10\sqrt{235}-50)$ feet from the base of the bridge.

II. *Solution by Frank Hawthorne, Columbia University.* As a consequence of the law of conservation of energy it can easily be shown that the velocity of a projectile at any point is that due to a fall from the directrix of the parabolic path. (See, *e.g.*, p. 206 of Jeans' Theoretical Mechanics.) Consider the plane of the projectile and let O_1 and O_2 be the edges of the bridge in this plane. The focus F of the parabola is somewhere on the vertical bisector of O_1O_2. The distance of the directrix above O_1 and O_2 is equal to the distance O_1F (or O_2F). By the opening sentence, in order to minimize the initial velocity, we must take the directrix as low as possible. This will occur when F is the midpoint of O_1O_2. We thus have the following construction:

With centers at O_1 and O_2 describe circles of 50 ft. radius, tangent to each other at F. F is the midpoint of O_1O_2. Draw the common tangent AB above the two circles. Draw a horizontal line DE parallel to O_1O_2 and 210 ft. below it. With center F and radius 260 ft. strike an arc intersecting DE at P. Draw PC perpendicular to AB. P is the point at which the arrow leaves the bow; F and AB are the focus and directrix of the parabolic path.

(1) The minimum initial velocity is the velocity of free fall from 260 ft., *i.e.*, $\sqrt{520g}=129$ ft./sec. approximately. ($g=32$ ft./sec^2.)

(2) The horizontal distance from P to a point directly beneath the center of the bridge is $\sqrt{260^2-210^2}=153$ ft. approximately.

(3) The initial direction of the arrow is along the bisector of angle FPC. The angle of elevation is arc sin $210/260+\frac{1}{2}$ arc cos $210/260=71°\ 56'$ approximately.

(4) The highest point on the trajectory is half way between F and the line AB, or 25 ft. above the bridge, or 240 ft. above the ground under the bridge.

Also solved by D. W. Alling, G. W. Grotts, Norman Miller, S. T. Parker, Herbert Reisman, A. Sisk, E. S. Smith, R. H. Urbano, and the proposer.

As a corollary to his method Hawthorne gave the following interesting construction. To find the path, for minimum initial velocity, which passes through two given points, simply divide the line joining the two points into two parts such that the lower part is h longer than the upper part, where h is the difference in height of the two points. This gives the focus F. The directrix is horizontal above the points at a distance from each equal to their respective distances from F.

The proposer furnished the following background for this problem. "The dimensions of the bridge are those of the Natural Bridge of Virginia. Some months ago Mr. Earl Chapin May, New York City author, on a visit to the Bridge related an incident recorded in his father's diary in the year 1858. According to Mr. May, his father was traveling with a road circus through this section in that year and states that one Charlie Tubbs, a strong man of the show, threw a stone over Natural Bridge. This statement immediately brought up the question of the physical possibility of such a feat."

A Property of an Odd Number of Consecutive Integers

E 692 [1945, 516]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider $2p+1$ consecutive integers. Show that: (1) the product of the first p of them diminished by the product of the last p gives an integer divisible by $p(p+1)$; (2) the product of the first p increased by the product of the last p gives an integer divisible by $p(p+1)$, unless $p+1$ is a prime factor of the remaining integer of the set.

(This is a generalization of problem 203 by Fitz-Patrick, p. 32 of G. de Longchamps' *Exercises d'arithmétique.*)

Solution by E. M. Beesley, University of Nevada. For $p \geqq 1$ and any integer k, let

$$A_k = \prod_{j=1}^{p} (k+j), \qquad B_k = \prod_{j=1}^{p} (k+p+1+j).$$

The following relations will be established:

(1) $A_k \pm B_k \equiv 0 \pmod{p}$

(2) $A_k - B_k \equiv 0 \pmod{p+1}$

(3) $A_k + B_k \equiv 0 \pmod{p+1}$, unless $p+1$ is an odd prime and $k \equiv 0 \pmod{p+1}$.

Since any collection of p consecutive integers must have a multiple of p as a factor, (1) is evident. If $k \not\equiv 0 \pmod{p+1}$, both A_k and B_k have a multiple of $p+1$ as a factor. Hence (2) and (3) are valid for such k. If $k \equiv 0 \pmod{p+1}$, then

$$k+j \equiv k+p+1+j \equiv j \pmod{p+1},$$

and consequently $A_k + B_k \equiv p! \pmod{p+1}$. This completes the verification of (2). Moreover

$$A_k + B_k \equiv 2(p!) \pmod{p+1},$$

and since $2(p!) \equiv 0 \pmod{p+1}$ if and only if $p+1$ is not an odd prime, this establishes (3).

Also solved by D. W. Alling, Murray Barbour, P. T. Bateman, B. A. Hausmann, Z. I. Mosesson, P. A. Pizá, E. P. Starke, and the proposer.

An Anharmonic Ratio in Space

E 693 [1945, 516]. *Proposed by N. A. Court, University of Oklahoma*

Through a given point to draw a line meeting three planes in three points so that the anharmonic ratio of the four points shall have a given value.

Solution by the Proposer. Let a line m through the given point P meet the two given planes α, β in the points Q, R, and let S be the point on m such that $(PQRS)=k$, where k is the given value of the anharmonic ratio.

The plane δ determined by the point S and the line $\alpha\beta$ cuts the third given plane γ along a line u. A line joining P to any point of u satisfies the conditions of the problem. The required lines thus form a flat pencil in the plane P-u and having P for center.

Limit of a Recursive Sequence

E 694 [1945, 516]. *Proposed by J. D. Bankier, McMaster University*

A sequence $\{x_n\}$ is defined recursively, in terms of two numbers x_0 and x_1, by the formula

$$x_n = \frac{n-1}{n}\, x_{n-1} + \frac{1}{n}\, x_{n-2}.$$

To what value does the sequence converge?

Solution by G. T. Williams, student, Harvard University. We write the equation as

$$x_n - x_{n-1} = -(x_{n-1} - x_{n-2})/n.$$

Then, by iteration,

$$x_n - x_{n-1} = (-1)^n(x_0 - x_1)/n!.$$

Summing both sides,

$$x_n - x_0 = (x_0 - x_1)\sum_{1}^{n} (-1)^n/n!$$

and

$$x_n \to x_1 + (x_0 - x_1)/e.$$

Also solved by D. W. Alling, Murray Barbour, P. T. Bateman, E. M. Beesley, Barney Bissinger, Harley Flanders, N. G. Gunderson, J. E. Hanson, J. B. Kelly, H. A. Luther, Norman Miller, Leo Moser, Z. I. Mosesson and M. J. Singer (jointly), S. T. Parker, A. S. Peters, E. P. Starke, Morgan Ward, J. G. Wendel, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4202. *Proposed by Vladimir Karapetoff, New York, N. Y.*

In a certain game of chance, consecutive numbers from 1 to n are written on a table. The same number n of discs are provided with consecutive numbers written on them. The discs are turned with the numbers down so that the players cannot see the numbers written on them. A player covers all the numbers on the table with the discs at random, because he does not see the numbers on them. The discs are then turned over and the score is made on the basis of the number of discs whose numbers agree with the numbers on the table which they are covering. It is required to deduce an expression for the chance that k of the n discs covered the correct numbers.

4203. *Proposed by N. J. Fine, Indianapolis, Ind.*

This is a generalization of E 651 [1945, 42]. If one is allowed n weighings on a beam balance, what is the maximum number A_n of coins, exactly one of which is bad, from which one can isolate the bad coin and determine whether it is heavy or light? (Cf. E 712 [1946, 156].)

4204. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The four products of the three sides of faces of a tetrahedron $ABCD$ are proportional to the tangents of the half angles of the corresponding cones of revolution inscribed in the trihedral angles at the vertices.

4205. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a right triangle has sides of integral lengths and the sum of the sides forming the right angle is a square, then the sum of the cubes of these two sides is the sum of two squares. Can the hypotenuse be a square?

SOLUTIONS

Feuerbach Points

3945 [1940, 115]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If in a triangle the distances d_1, d_2, d_3 of the midpoints of the sides of the triangle to a tangent to the nine-point circle satisfy a relation $\sqrt{d_1} \pm \sqrt{d_2} \pm \sqrt{d_3} = 0$, the point of contact of that tangent is one of the Feuerbach points.

Solution by R. Goormaghtigh, Bruges, Belgium. According to Godt's theorem (*Münchener Berichte*, 1896), the Feuerbach points of a triangle are the four points of the nine-point circle such that the distance of any of them to the midpoint of one of the sides equals the sum of its distances to the midpoints of the two other sides.

But, if d_1 is the distance of the midpoint A_m of the side BC of a triangle ABC to the tangent at a point T to the nine-point circle, then

$$d_1 = \overline{TA}_m^2/R$$

R being the circumradius of ABC. The proposed theorem is therefore another form of Godt's theorem.

Infinite Series and Products Deranged

4138 [1944, 533]. *Proposed by G. Pólya, Stanford University*

Given two positive integers p and q define

$$S_{p,q} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2p} - \frac{1}{3} - \frac{1}{5} - \cdots - \frac{1}{2q+1} + \frac{1}{2p+2} + \cdots + \frac{1}{4p} - \frac{1}{2q+3} - \cdots - \frac{1}{4q+1} + \cdots,$$

$$P_{p,q} = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2p}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots \cdot\left(1-\frac{1}{2q+1}\right)\left(1+\frac{1}{2p+2}\right)\cdots.$$

We obtain the series in which blocks of p positive terms alternate with blocks of q negative terms by rearranging the terms of the well known series

$$\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \cdots = S_{1,1} = 1 - \log 2,$$

and the product by rearranging correspondingly

$$(1 + \tfrac{1}{2})(1 - \tfrac{1}{3})(1 + \tfrac{1}{4})(1 - \tfrac{1}{5}) \cdots = P_{1,1}.$$

Show directly that

$$P_{p,q} = (p/q)^{1/2}$$

and hence derive the well known result

$$S_{p,q} - S_{1,1} = \log (p/q)^{1/2}.$$

Solution by the Proposer. 1. Define

$$F_n = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\cdots\left(1+\frac{1}{2n}\right).$$

Then, by Wallis' formula, or Stirling's formula,

$$F_n = \frac{3}{2}\,\frac{5}{4}\,\frac{7}{6}\cdots\frac{2n+1}{2n} = \frac{(2n+1)!}{(n!2^n)^2} = \frac{2n+1}{2^{2n}}\binom{2n}{n} \sim \frac{2n^{1/2}}{\pi^{1/2}}$$

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots\left(1-\frac{1}{2n+1}\right) = \frac{2}{3}\,\frac{4}{5}\,\frac{6}{7}\cdots\frac{2n}{2n+1} = \frac{1}{F_n}$$

and the partial product of $P_{p,q}$ with subscript $(p+q)n$ is

$$\frac{F_{pn}}{F_{qn}} \sim \left(\frac{pn}{qn}\right)^{1/2}.$$

This last limit can also be found by more elementary methods, without using Wallis' or Stirling's formula.

2. The series

$$\log P_{1,1} - S_{1,1} = [\log(1+\tfrac{1}{2}) - \tfrac{1}{2}] + [\log(1-\tfrac{1}{3}) + \tfrac{1}{3}] + [\log(1+\tfrac{1}{4}) - \tfrac{1}{4}] + \cdots$$

is absolutely convergent since

$$|\log(1-x)+x| = \left|-\frac{x^2}{2}-\frac{x^3}{3}-\cdots\right| < \frac{|x|^2}{2(1-|x|)} \leqq |x|^2$$

for $|x| \leqq 1/2$. The value of an absolutely convergent series is not altered by a rearrangement of the terms, and so

$$\log P_{p,q} - S_{p,q} = \log P_{1,1} - S_{1,1}.$$

Editorial Note. The functions F_n and $1/F_n$ are related to the definite integrals

$$I_m = \int_0^{\pi/2} \sin^m x\,dx, \qquad I_0 = \pi/2, \qquad I_1 = 1,$$

where m is an integer greater than unity. Integration by parts gives

$$I_m = \frac{m-1}{m}\,\frac{m-3}{m-2}\cdots\frac{m-(2k-1)}{m-2(k-1)}\,I_{m-2k}.$$

From the relations $I_{2m+1} < I_{2m} < I_{2m-1}$, we find that

$$\lim_{m\to\infty} I_{2m}/I_{2m-1} = 1.$$

We now have

$$I_{2np} = F_{np}\frac{I_0}{2np+1}, \qquad I_{2np-1} = \frac{1}{F_{np}}\frac{2np+1}{2np},$$

$$I_{2np}/I_{2np-1} = (F_{np})^2 I_0\, 2np/(2np+1)^2,$$

$$\frac{I_{2np}/I_{2np-1}}{I_{2nq}/I_{2nq-1}} = \left(\frac{F_{np}}{F_{nq}}\right)^2 \frac{p}{q}\left(\frac{2nq+1}{2np+1}\right)^2,$$

$$P_{pq} = \lim_{n\to\infty}\frac{F_{np}}{F_{nq}} = \sqrt{\frac{p}{q}}. \tag{1}$$

In order to derive Wallis' formula we consider

$$I_{2m}/I_{2m-1} = I_0/(I_{2m-1})^2 2m$$

$$= I_0\frac{1\cdot 3}{2\cdot 2}\frac{3\cdot 5}{4\cdot 4}\cdots\frac{(2m-3)\cdot(2m-1)}{2(m-1)\cdot 2(m-1)}\frac{2m-1}{2m}.$$

Hence

$$\frac{\pi}{2} = \prod_{m=1}^{\infty}\frac{2m\cdot 2m}{(2m-1)\cdot(2m+1)}.$$

See Goursat-Hedrick, *Mathematical Analysis*, vol. 1, p. 240.

An elementary proof of (1) not involving I_m is as follows. It will suffice to consider $p>q$, then by reduction we have

$$F_{np}/F_{nq} = \prod_{j=1}^{n(p-q)}\left[1+\frac{1}{2(j+nq)}\right],$$

$$\log(F_{np}/F_{nq}) - \sum_{j=1}^{n(p-q)} 1/2(j+nq)$$

$$= \sum_{j=1}^{n(p-q)}\left\{\log\left[1+\frac{1}{2(j+nq)}\right] - \frac{1}{2(j+nq)}\right\}.$$

The absolute value of the summand $\{\cdots\}$ is less than $1/8(1+nq)^2$, and hence the absolute value of the last sum is less than $n(p-q)/8(1+nq)^2$. Thus

$$\lim_{n\to\infty}\log\left(\frac{F_{np}}{F_{nq}}\right) = \frac{1}{2}\lim_{n\to\infty}\sum_{j=1}^{n(p-q)}\frac{1}{j+nq}.$$

In order to find this limit, divide the interval on the x axis from the origin to $(p-q)/q$ into $n(p-q)$ equal subintervals of length $1/qn$, and at the end of the jth interval erect the ordinate $1/(j+nq)$. We now have

$$\lim_{n\to\infty}\sum_{j=1}^{n(p-q)} 1/(j+nq) = \int_0^{(p-q)/q}\frac{dx}{1+x} = \log\left(\frac{p}{q}\right).$$

Hence

$$P_{pq} = \lim_{n\to\infty}\frac{F_{np}}{F_{nq}} = \sqrt{\frac{p}{q}}.$$

A direct proof of the formula for S_{pq} is as follows. Again taking $p>q$ set

$$T_{np} = \sum_{j=1}^{np} \frac{1}{2j}, \qquad T_{nq} = \sum_{j=1}^{nq} \frac{1}{2j+1},$$

Then

$$T_{np} - T_{nq} = \sum_{j=1}^{nq} \left(\frac{1}{2j} - \frac{1}{2j+1} \right) + \frac{1}{2} \sum_{j=1}^{n(p-q)} \frac{1}{j+nq},$$

$$\lim_{n\to\infty} (T_{np} - T_{nq}) = S_{11} + \log \sqrt{p/q},$$

$$S_{pq} = S_{11} + \log \sqrt{p/q}, \tag{2}$$

the log term being derived as before, and the terms of $T_{np}-T_{nq}$ may be rearranged so that we have first the first p positive terms of S_{11}, then next the first q negative terms, the next set of p positive terms, and so on exhausting all of the $n(p+q)$ terms.

Two Cubes and a Square

4140 [1944, 534]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find an integer of two digits such that the difference between its cube and that of the integer whose digits are those of the first in reversed order is a perfect square.

Solution by Monte Dernham, San Francisco. We have

$$(10t + u)^3 - (10u + t)^3 = A^2. \tag{1}$$

This may be written

$$3(t - u)[37(t + u)^2 - 27tu] = (A/3)^2. \tag{2}$$

Hence either $t+u$ or $t-u$ is a multiple of 3, which serves to eliminate from consideration all but 16 pairs of two digit integers. Further discarding those the difference of whose cubes terminates in 2, 3, 7 or 8, we are restricted to one or more of the following integers

$$87, 82, 81, 74, 72, 63, 54, 51, 42, \tag{3}$$

an examination of which discloses a single solution,

$$74^3 - 47^3 = 549^2.$$

Solved also by D. H. Browne, M. L. Constable, B. A. Hausmann, Irving Kaplansky, J. B. Kelly, A. D. Maxwell, W. C. Rufus, E. P. Starke, R. H. Wilson, Jr. and the proposer.

Editorial Note. It will be seen that the case $u=0$ must be rejected since 37 is a prime, and this is also easily seen without using this primality. Tests of the numbers in (3) can be easily applied without use of tables for cubes or squares as follows. We take outside the $[\cdots]$ in (2) the common factors of the two terms inside leaving the remaining factor c. Then we have for the two factors:

87, 3^3, $c \equiv 1 \bmod 3$; 82, $3^2 2^3$, $c \equiv 1 \bmod 2$; 81, $3^4 7$, $c \equiv 5 \bmod 7$;
74, 3^2, $c = 61^2$; 72, $3^4 5$, $c \equiv -3 \bmod 5$; 63, 3^6, $c = 31$;
54, 3^4, $c = 13 \cdot 7$; 51, $3^3 4$, $c \equiv 1 \bmod 3$; 42, $3^2 2^3$, $c = 31$.

Hence the only solution is for 74 as stated above. A later note from Dernham eliminated from (3) the cases 87, 51, 42.

Classes of Infinite Series

4142 [1944, 593]. *Proposed by G. Pólya, Stanford University*

Find a sequence of real numbers $a_1, a_2, a_3, \cdots$ so that $\sum_1^\infty a_n$ converges, $\sum_1^\infty a_n^3$ diverges, $\sum_1^\infty a_n^5$ converges. More generally, let C be an arbitrarily given (finite or infinite) class of positive integers. There exists a sequence of real numbers $a_1, a_2, \cdots, a_n, \cdots$ adapted to C so that, for $l = 1, 2, 3, \cdots$, the series

$$a_1^{2l-1} + a_2^{2l-1} + \cdots + a_n^{2l-1} + \cdots$$

converges or diverges according as l does or does not belong to C.

Solution by N. J. Fine, Indianapolis, Ind. For a given positive odd integer w we shall construct a sequence of real numbers c_i with the following properties:

(1) $$S_w(k) \equiv \sum_{i=1}^{\infty} c_i^k \quad \text{converges for all positive odd } k \neq w,$$

(2) $$S_w(w) \equiv \sum_{i=1}^{\infty} c_i^w \quad \text{diverges.}$$

Throughout this discussion k will be restricted to be a positive odd integer.

The c_i will be constructed in blocks $B_n(w)$ of the form

$$\begin{aligned} B_n(\omega) = {} & A_1 n^{-1/w}, \cdots, A_1 n^{-1/w} && (k_1 \text{ terms}) \\ & A_2 n^{-1/w}, \cdots, A_2 n^{-1/w} && (k_2 \text{ terms}) \\ & \cdots\cdots\cdots\cdots\cdots\cdots && \cdots\cdots \\ & A_r n^{-1/w}, \cdots, A_r n^{-1/w} && (k_r \text{ terms}), \end{aligned}$$

where $r = (w+1)/2$ and the A_i and k_i are to be determined subject to the conditions that

(1) k_i are positive integers

(2) $$|A_i| \leqq 1, \qquad A_i \neq A_j \quad \text{if} \quad i \neq j$$

(3) $$\sum_{i=1}^{r} k_i A_i^k = 0 \quad \text{if} \quad k < w$$

(4) $$\sum_{i=1}^{r} k_i A_i^w = \Delta \neq 0.$$

These conditions may be satisfied by first choosing the A_i as any set of distinct rationals not greater than unity, solving the system of equations (3) and (4) for k_i in the form $k_i=\Delta D_i/D$, where D is the determinant of the system, choosing Δ as a common multiple of all the denominators of D_i/D expressed as fractions, and finally changing the signs of all the negative k_i by changing the signs of the corresponding A_i. These steps are all possible since the D_i and D are Vandermonde determinants and hence are non-zero rationals. With conditions (1) $\cdots$ (4) satisfied it is easy to verify that the required properties 1 and 2 belong to the sequence c_i.

Now, given an arbitrary class M of positive odd integers $\{w_j\}$, we construct a series $S(k)$ which diverges if $k\in M$ and converges otherwise. Let C_j be a bound for the partial sums of $S_{w_j}(k)$ for all $k\neq w_j$; this bound exists since the $|A_i|$ were chosen ≤ 1, and $S_{w_j}(k)$ converges absolutely and uniformly for $k>w_j$. Now write down the double series in which the terms of the jth row are $2^{-j}C_j^{-1}$ multiplied by the terms of $S_{w_j}(k)$, and sum this series by diagonals to give the required series. As an example we construct the series S_3:

$$1^{-1/3}-\tfrac{1}{2}\cdot 1^{-1/3}-\tfrac{1}{2}\cdot 1^{-1/3}+2^{-1/3}-\tfrac{1}{2}\cdot 2^{-1/3}-\tfrac{1}{2}\cdot 2^{-1/3}+\cdots$$
$$+n^{-1/3}-\tfrac{1}{2}n^{-1/3}-\tfrac{1}{2}n^{-1/3}+\cdots.$$

Eight-Digit Squares

4144 [1944, 593]. *Proposed by V. Thébault, Tennie, Sarthe, France*

For what base is a number of eight digits of the form *ababcdcd* the square of a four digit number *mnmn*, given that *a b c d m n* is a permutation of six consecutive digits?

Solution by E. P. Starke, Rutgers University. Let R be the base and, for brevity, put $A=aR+b$, $C=cR+d$, $M=mR+n$. We have the

$$AR^6+AR^4+CR^2+C=M^2(R^2+1)^2$$

or

$$AR^2-A+(A+C)/(R^2+1)=M^2.$$

Now the fact that $0<A+C<2R^2$ implies that the integer $(A+C)/(R^2+1)$ is unity. Thus we have

$$AR^2-A+1=M^2, \qquad A+C=R^2+1.$$

Adopting the following procedure, we may easily test any proposed value of R: determine the values of M for which

$$A=(M^2-1)/(R^2-1) \tag{1}$$

is an integer, find $C=R^2+1-A$, and compare the resulting digits.*

For $M<R^2$, R^2-1 divides M^2-1 for $M=1$, R, R^2-R-1, R^2-2. The corre-

* There must be no duplicate digits, and the difference between the greatest and smallest of the six digits must be 5.

sponding values of A are 0, 1, R^2-2R, R^2-3; and $C=R^2+1$, R^2, $2R+1$, 4. The first two sets must be discarded because $C \geqq R^2$. For the last two we easily compute $m=R-2$, $R-1$; $n=R-1$, $R-2$; $a=R-2$, $R-1$; so that in each case there are duplicate digits. Furthermore, $M^2-1 \equiv 0 \pmod{R^2-1}$ has no other solutions than the values of M listed above, when $R-1$ and $R+1$ are primes or powers of primes (thus there is no solution for $R=10$.)

If R is odd we have, besides the above, the following values:

$$
\begin{aligned}
M &= \tfrac{1}{2}(R^2-2R-1), \quad \tfrac{1}{2}(R^2+2R-1), \quad \tfrac{1}{2}(R^2-3), \quad \tfrac{1}{2}(R^2+1);\\
A &= \tfrac{1}{4}(R^2-4R+3), \quad \tfrac{1}{4}(R^2+4R+3), \quad \tfrac{1}{4}(R^2-5), \quad \tfrac{1}{4}(R^2+3);\\
C &= \tfrac{1}{4}(3R^2+4R+1), \quad \tfrac{1}{4}(3R^2-4R+1), \quad \tfrac{3}{4}(R^2+3), \quad \tfrac{1}{4}(3R^2+1).
\end{aligned} \tag{2}
$$

When reduced to base R, the digits of the numbers in the first column above are

$$m=\tfrac{1}{2}(R-3),\ n=\tfrac{1}{2}(R-1),\ a=\tfrac{1}{4}(R-5),\ b=\tfrac{1}{4}(R+3),\ c=\tfrac{1}{4}(3R+1),\ d=\tfrac{1}{4}(3R+1);$$

$$m=\tfrac{1}{2}(R-3),\ n=\tfrac{1}{2}(R-1),\ a=\tfrac{1}{4}(R-7),\ b=\tfrac{3}{4}(R+1),\ c=\tfrac{3}{4}(R+1),\ d=\tfrac{1}{4}(R+1);$$

according as $R=4x+1$ or $4x-1$. When a similar break-down is made for the other values of M, A, C, it is easily seen that the difference between the smallest and greatest digits exceeds 5 whenever $R>9$. Obviously $R>5$ if there exist six consecutive digits. Trying $R=7$, 9 we obtain just two solutions with no duplicate digits, both of which come from the fourth column of (2) and in which A, M, C form an arithmetic progression. The solutions are

$$\text{base } 7,\ 16165252 = (3434)^2,$$

$$\text{base } 9,\ 23236767 = (4545)^2.$$

For other solutions, if they exist, R^2-1 must admit further factorization than appears above, but with larger R and increasing range of digits the chance diminishes that all six digits be consecutive. Further trials have yielded no more solutions, and a general treatment seems out of the question.

Malfatti Problem

4145 [1945, 47]. *Proposed by Howard Eves, College of Puget Sound*

Find the positions of three non-overlapping circles in a triangle which have a maximum combined area.

Remarks by the Proposer. In 1803 Malfatti proposed the problem of cutting three circular cylindrical holes out of a solid right triangular prism in such a way that the cylinders and the prism have the same altitude and that the combined volumes of the cylinders be a maximum. Malfatti intuitively reduced this problem to another, now commonly referred to as "Malfatti's problem": To inscribe within a triangle three circles each of which shall be tangent to the other two and to two sides of the triangle. This reduced problem has enjoyed a long history and has been solved and generalized in many ways. Apparently

Malfatti considered this reduced problem as equivalent to the original problem. That such is not the case was indicated in 1929, when Lob and Richmond pointed out that in the case of the equilateral triangle the inscribed circle and two of the smaller circles that can be fitted into the corners have a combined area which is greater than that given by the Malfatti arrangement, the ratio of the areas of the two respective sets of circles being $22(2+\sqrt{3})/81>1$.

That the Malfatti arrangement certainly cannot always lead to a maximum combined area is more easily and more strikingly seen by considering an isosceles triangle with unit base and very long altitude, h, on this base. Then, letting A be the combined area of the Malfatti circles, we have

$$\lim_{h\to\infty} A = 3\pi/8.$$

On the other hand, if A' is the combined area of three circles fitted one above the other and tangent to the legs of the triangle, then

$$\lim_{h\to\infty} A' = 3\pi/4.$$

Clearly, then, the original and the reduced Malfatti problems are not equivalent, and the problem stated at the head of these remarks is raised. The analogous problem of finding the positions of four non-overlapping spheres inside a tetrahedron and having a maximum combined volume is also raised.

Note. Solutions of 4145 are still invited.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

A conference on algebra will be held at the University of Chicago during the week of July 15–19, 1946. Some of the sessions will emphasize the recent contacts of algebra with other branches of mathematics such as topology, function theory, and geometry. It is hoped that the meetings will be held in Eckhart Hall. Reservations should be made directly with Chicago hotels. Hotels in the vicinity of the University of Chicago are the Broadview, 5540 South Hyde Park Blvd.; Del Prado, 5307 South Hyde Park Blvd.; Shoreland, 5454 South Shore Drive, Windermere, 1642 East 56th Street; Mira-Mar, 6218 South Woodlawn. Because of the acute shortage of hotel space, reservations should be made as soon as possible.

Dr. Theodore Hailperin has been appointed to an assistant professorship at Lehigh University.

W. I. Layton of Amarillo College has been appointed Professor of Mathematics and Head of the Department at Austin Peay State College, Clarksville, Tennessee.

Dr. P. V. Reichelderfer of Ohio State University and Assistant Professor George Whaples of the University of Pennsylvania have been appointed to assistant professorships at the University of Wisconsin.

Dr. M. F. Smiley has returned to Lehigh University as an associate professor.

The following appointments to instructorships are announced:
Michigan State College: Stanley Johnson, Mrs. Madelyn Kintner, Miss J. E. Master, O. T. McMillan, Nicholas Musselman, Frank Saidel.
University of Houston: Dr. Albert Newhouse

Associate Professor Emeritus Joseph Allen of the College of the City of New York died March 4, 1946. He was a charter member of the Association.

Professor Emeritus Arthur Korn of Stevens Institute of Technology died December 21, 1945.

Professor Emeritus T. A. Pierce of the University of Nebraska died August 18, 1945.

R. G. M. Sabel of Bristol High School, Bristol, Connecticut, died February 24, 1946.

Professor Althéod Tremblay of Laval University, Quebec, died March 1, 1946.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1946:

The Catholic University of America. From July 1 to August 8 the following advanced courses will be offered: By Professor Finan: theory of equations, fundamentals of mathematics. By Professor Ramler: advanced Euclidean geometry, synthetic projective geometry, differential equations. By Professor Rice: solid analytic geometry, calculus of observations.

Teachers College, Columbia University. From July 8 to August 16 the following courses in the teaching of mathematics will be offered: By Professor Fawcett: a one-week course on the teaching of geometry. By Professor Fehr: teaching algebra in secondary schools, professionalized subject matter in advanced secondary mathematics. By Dr. Lazar: teaching geometry in secondary schools, history of mathematics. By Mr. Mirick: elementary mechanics (statics), observation and participation in the teaching of geometry. By Professor Reeve: teaching and supervision of mathematics—junior high school, teaching and supervision of mathematics—senior high school. By Professor Schlauch: modern business arithmetic. By Professor Shuster: teaching arithmetic in elementary school,

field work in mathematics. In addition, on consecutive Thursdays beginning on July 11, there will be five special lectures and discussions relating to the reorganization and teaching of mathematics in the post-war world.

The University of Southern California. From June 24 to August 2 the following advanced courses will be offered: By Dr. Hammer: calculus of observations. By Professor Hyers: theory of functions of a complex variable. By Professor Snapper: theory of equations and determinants, matrix theory. By Professor Steed: differential equations. By Professor White: introduction to higher geometry. By Professor Ulam: theory of probability and statistics.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

IMPORTANT LEGISLATION BEFORE CONGRESS

For over a year Congress has been debating various bills proposing the creation of a National Science Foundation or a National Research Board for the encouragement of a more extensive, national program of scientific research. The first bill to appear was S. 825, introduced by Senator Byrd on April 4, 1945. It proposed the establishment of a Research Board for National Security as an independent government agency. A somewhat similar bill was introduced into the House by Representative May on June 11, 1945, but in that measure it was proposed that the Research Board be set up by the National Academy of Sciences in cooperation with the War and Navy Departments. On July 9, 1945, Senator Fulbright proposed S. 1248, which would set up within the Department of Commerce a Bureau of Scientific Research. The proposed Bureau would absorb the Office of Production Research and Development of the War Production Board and the National Inventors Council. On July 19, Senator Magnuson introduced S. 1285, which provided for the creation of a National Science Foundation, and generally followed the major recommendations of Vannevar Bush's Report to the President. Four days later Senator Kilgore introduced S. 1297, which differed from S. 1285 in proposing that a National Science Foundation be administered by a Director rather than by a Board of unpaid, part-time scientists. In addition, S. 1297 included the social sciences within the proposed program and provided for some new patent regulations.

Approximately 100 scientists and a few laymen participated in hearings upon the various bills during August, September, and October, 1945. Opinions were obtained from university, industrial, and government scientists, as well as from agencies and businesses whose chief concern is the application of science. All

but one witness favored the establishment of a National Science Foundation.

The president of the National Academy of Sciences opposed the creation of a National Science Foundation, as proposed, partly because of his belief that politics could not be eliminated from the distribution of funds set aside for scientific research. A majority of witnesses favored the inclusion of the social sciences in the Foundation. A majority also seemed to favor the type of administration previously advocated in S. 1285. These hearings and later discussions led to the introduction of S. 1720 on December 21. This bill, introduced by Senator Kilgore, with Senators Johnson, Pepper, Fulbright, and Saltonstall as co-sponsors, embodied many improvements. However, it failed to gain the support of many influential scientists who continued to back the original Magnuson Bill. Representatives of the latter group were interviewed in January by a group from the Senate headed by Senators Kilgore and Saltonstall, and several modifications in the provisions of S. 1720 were considered.

As a result, another bill, S. 1850, was introduced into the Senate on February 21; the sponsors of this compromise measure were Senators Kilgore, Magnuson, Pepper, Johnson, Fulbright, Saltonstall, Ferguson, and Thomas. Some of the major provisions of the bill are briefly summarized as follows:

1. Establishes a National Science Foundation administered by an Administrator appointed by the President with the consent of the Senate and after consultation with the National Science Board.

2. Provides for the creation of a National Science Board to be made up of nine persons appointed by the President after obtaining the advice and consent of the Senate and the Chairmen of the several Divisional Scientific Committees.

3. Designates the following Divisions to be established within the Foundation: (a) Mathematical and Physical Sciences; (b) Biological Sciences; (c) Social Sciences; (d) Health and Medical Sciences; (e) National Defense; (f) Engineering and Technology; (g) Scientific Personnel and Education; (h) Publications and Information; and (i) additional divisions not to exceed three in number. Each Divisional Committee, except National Defense, shall consist of 5 to 15 members appointed by the Administrator with the advice and approval of the Board. The term of service of each member shall be three years.

4. Provides that "The Board shall continuously survey the activities and management of the Foundation, and shall periodically evaluate the achievements of the Foundation in accomplishing the objectives of this Act. Each divisional scientific committee shall survey continuously the scientific field which it encompasses, shall undertake to determine the specific scientific needs of such field, and shall evaluate proposed programs and projects." "Until the Administrator and the Board have received general recommendations from the Division of Social Sciences regarding the support of research through that Division, support of social science research shall be limited to studies of the impact of scientific discovery on the general welfare and studies required in connection with other projects supported by the Foundation."

5. Authorizes the awarding of scholarships and fellowships "in any field of science, including but not limited to the mathematical, physical, biological, medical, and social sciences at nonprofit institutions of higher education, or other institutions, selected by the recipient of such aid, for such periods as the Administrator may determine, in the United States or in foreign countries."

6. Establishes a national register of scientific personnel.

7. Provides for the use and dissemination of research findings.

8. Authorizes the head of any Government agency, "with the approval of the President and through the Department of State, to conclude reciprocal agreements with foreign governments or agencies thereof, relating to the interchange of scientific and technological information, and the use and availability of patents and patent rights owned or controlled by the respective governments." "The Administrator may defray the expenses of representatives of Government agencies and other organizations and of individual scientists to accredited international scientific congresses and meetings whenever he deems it necessary in the promotion of the objectives of this Act.

Just as it appeared that an acceptable bill had finally been written, two new science bills were introduced. One by Congresswoman Luce (H.R. 5332) would create a Department of Science and Health with a Secretary in the Cabinet. The Willis Bill, S. 1777, proposes that the President appoint a self-perpetuating committee of 50 from a list of nominees submitted by the National Academy of Science for the purpose of administering a program of scientific activity. This latter bill is receiving support from the minority of scientists who have opposed the creation of a National Science Foundation because of its possible political implications.

EDUCATION FOR ALL AMERICAN YOUTH: A REVIEW

H. E. WAHLERT, New York University

Teachers of college mathematics become aware of the results of secondary education, sometimes painfully, when they meet these results in their freshman classes. It may then be too late to suggest changes that would lead to better results. Thus it seems desirable for college mathematics teachers to be aware of the opinions, plans, and ideas, as expressed by leading educators in the field of secondary education, before they are put into effect. They can then be evaluated from the point of view of their probable effect on our future students; we can give support to those we find desirable and oppose those we find undesirable. The Mathematical Association of America has long been interested in educational questions and has recently expressed this interest, even more definitely, through the appointment of a Coordinating Committee "to keep in close touch with all educational movements in the United States and Canada.* Thus, following the aim of this committee, it is the purpose of this article to discuss the attitude toward mathematics of one of the many recent publications professing

* This MONTHLY, vol. 52, May, 1945, p. 294.

to describe a desirable secondary education, *Education for All American Youth.**

Two hypothetical schools, one in "Farmville" and one in "American City", are described in Chapters 2–4 as if they had been in existence for five years following the end of the war. These schools are planned for all youth in grades 10 to 14. In junior high school, grades 7–9, these youth will have had a curriculum which, "in its broad outlines," is "the same for all pupils." An analysis of the common qualities and differences of youth leads to a broad point of view toward education. From this point of view evolve the curricula of the two schools. Since they are basically similar, the curriculum of American City can serve as an example. The program is divided into two areas: "Common Studies" and "Differential Studies." All students, in heterogeneous groups, take the work in Common Studies. The differences in the interests of students, for example, occupational, recreational, and so on, are determined, and, by suitable guidance and planning, the student and his advisers map out his program in the field of Differential Studies. The material studied in the area of Common Studies includes health and physical education, a year course whose primary aim is to "understand the social significance of science," and "a continuous course for all, planned to help students grow in competence as citizens of the community and the nation; in understanding of economic processes and of their roles as producers and consumers; in cooperative living in family, school, and community; in appreciation of literature and the arts; and in the use of the English language."

The average number of periods per day allotted to these divisions in the grades 10–14 is given in the table below:

	10	11	12	13	14
Common Learnings	2	2	2	1	1
Science	1	0	0	0	0
Health and Physical Education	1	1	1	1	1
Vocational Preparation	1	2	2	3	3
Individual Interests	1	1	1	1	1

Since this report deals with education in grades 10–14, mathematics, as such, is not required of all students.† It is assumed that, in the course of study through junior high school, "mastery over the processes and principles" of mathematics "which everyone needs to know" has been developed. However, it is recognized

* National Education Association and American Association of School Administrators, Educational Policies Commission, Education for All American Youth, Washington, D. C., the Commission, 1944. Quotations in this article, not otherwise footnoted, are from this book.

† This is not inconsistent with the opinion expressed in the final report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, Bureau of Publications, Teachers College, Columbia University, New York 1940, p. 36.

that some students may not have developed this mastery, and so, "if a student lacks that mastery, remedial instruction is given in the tenth grade, longer if necessary" in American City. In Farmville the school operates "a mathematics workshop with a teacher in charge at all times, where remedial instruction is given and where any student may go at any time for help with the mathematical operations which he needs to use."

In the high schools, mathematics proper follows the line of vocational or occupational interests and "the amount of formal instruction may vary from nothing to three full years of systematic study." Students who plan to enter trades in which some mathematics is used would learn their mathematics in the trade course at the time that its use arises. Students who plan to enter such an occupation as engineering, in which much formal mathematics is needed, would take the necessary mathematics, probably in sequential courses, in part of the time set aside in their program for "Vocational Education."

Throughout the report primary emphasis is placed on "education in the responsibilities and privileges of citizenship." However, the ability to think rationally is also one of the main goals of the proposed plan. It is of interest to note that although mathematicians have often felt that "the chief end of mathematical study must be to make the pupil think,"* nowhere is it proposed that any use be made of mathematics in contributing to the general student's ability to think rationally.

Teachers of college mathematics are especially interested in the "boy who enjoys mathematics for its own sake" and in "Philip Scott, who solves problems in calculus for the fun of it." What do the secondary schools of Farmville and American City offer to these boys? First, they would take, along with all their fellow students, including Johnnie Lowique, the courses in the area of Common Studies. In these classes, with heterogeneous grouping, students with differing talents could offer their different abilities to the learning of the group. Thus Philip Scott and Johnnie Lowique would come to appreciate the talents of each other. In the area of Differential Studies, Philip Scott would probably choose sequential courses in mathematics for his work in the division of Vocational Preparation, and might pursue further studies in mathematics in the division of Individual Interests. He would "be encouraged to work well beyond the average of the class"; he would be able to pursue his interest within the regular classes or as an individual or in a small group, going as rapidly and as far as possible, and he would be "allowed extra time to pursue his own interests." He could get additional assistance from correspondence courses from the extension divisions of universities and colleges. In his study of mathematics the teachers would have acquainted him "with the nature and requirements of scientific and engineering occupations, as well as—the uses of mathematics and science in other occupations." In the course in science, required of all students, he will have learned "that most scientific advances have depended upon precise measurement and accurate calculation and that mathematics is indispensable to scientific inquiry."

* Young, J. W., Lectures on Fundamental Concepts of Algebra and Geometry, New York, The Macmillan Company, 1911, p. 4.

His interest in mathematics might lead him to go on the field trip in which dams and power projects are studied at firsthand and thus he will see further "that knowledge of mathematics and physics is essential for the engineer." He will learn, perhaps from the other students with related interests, (he may not look into it himself, since he does mathematics "for the fun of it"), that mathematics and science "must *stand up to practical testing*" by applying them "to gasoline motors, electrical equipment, crop production, food preparation and health."

What would happen in the high schools described in *Education for All American Youth* to "the stupid boy, the one who did not like school, the one who liked to play with little wind wheels, who liked to fight, who actually did run away from school, and who stood near the bottom of the class in mathematics?"* It is of course the hope of the authors of the report that the potential abilities of a Newton would be recognized with the attention these schools give to guidance, and that the schools themselves, with their opportunities for the pursuit of individual interests, would so appeal to a Newton that he would not run away from school. It is recognized, however, that a pupil may be "working far below his ability." In this event "the problem is one to be approached through counseling"; however, "neither skilful guidance nor good teaching—is a panacea. When counselors and teachers have done their best, a student may yet persist in doing far less than he is able to do. When this occurs, the school must enter the fact on the student's record, while it continues its search for causes and cures."

The report is well aware of the possible effects which the poverty of the environment may have on the mind of a child. It is pointed out that "it is possible for a boy or girl to grow up in this community or any other, and have his personality dwarfed or distorted because of the poverty of his environment. Not only is that possible, but it happens, year in and year out, to thousands of children." It is of course the aim of the authors of the report to have the school environment so rich and interesting that, even though the curriculum is organized for *all* American youth, Philip Scott or even an Isaac Newton would be challenged and would realize his potentialities. In the attempt to develop an education for *all* American youth, the question arises as to whether the education proposed has "been largely developed in the interests of the weaker students" and whether difficulties have "been placed in the path of the superior student which—make it difficult for him to obtain the education of which he is capable?" Will the gifted student find these schools challenging, or will they be "wasteful of a talented student's precious years of youth"?†

The report closes with the question: "Would you like your children to attend schools like those of Farmville and American City?" The question of interest to college teachers of mathematics is whether we would like to have our children and the children who grow up to be our students attend such schools?

* Sir Isaac Newton as described in Smith, D. E., Why should not all mathematics be elective?, Mathematics Teacher, vol. 38, December, 1945, p. 360.

† The last three quotations are from: MacDuffee, C. C., An objective in education, This MONTHLY, vol. 52, August-September, 1945, pp. 359, 361.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE MICHIGAN SECTION

The twenty-second annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan in Ann Arbor on Saturday, March 18, 1944. This meeting also constituted the meeting of the Mathematics Section of the Michigan Academy of Science, Arts, and Letters. Morning and afternoon program sessions were held in addition to a luncheon-business meeting. The Chairman of the Section, Professor G. G. Speeker, presided at all sessions.

There were fifty-one people in attendance, including the following thirty-six members of the Association: J. W. Bradshaw, J. B. Brandeberry, Father Richard Cebula, C. J. Coe, A. H. Copeland, C. C. Craig, Wayne Dancer, P. S. Dwyer, C. M. Erikson, F. D. Faulkner, Peter Field, K. W. Folley, J. S. Frame, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, P. S. Jones, Wilfred Kaplan, L. C. Karpinski, C. E. Love, Sister M. Mercedes, A. L. Nelson, E. A. Nordhaus, J. E. Powell, G. Y. Rainich, C. C. Richtmeyer, L. J. Rouse, T. R. Running, E. R. Sleight, G. G. Speeker, R. M. Thrall, Fern Welker, E. T. Welmers, R. L. Wilder.

Professor J. W. Bradshaw of the University of Michigan was elected Chairman for the following year, and Professor C. J. Coe of the University of Michigan was elected Secretary-Treasurer.

The following program of eight papers was presented:

1. *Prediction of the spacings in a certain molecule*, by Dr. Wilfred Kaplan, University of Michigan.

Professor K. Fajans has raised the question as to how much information concerning the spacings of the atoms in a molecule can be obtained from consideration of the Coulomb forces alone. In this paper Dr. Kaplan answered the question for the molecule P_4O_6 in which the Coulomb potential energy V is regarded as a function of a single parameter t determining the spacings of the oxygen atoms. He computed the value of t for a minimum value of V, obtaining a value in fair agreement with the experimental result, and offered an explanation of the discrepancy.

2. *A type of transformation of plane curves*, by Professor V. G. Grove, Michigan State College.

With a given source of light, each plane curve as a reflector is associated with another plane curve as its caustic. Professor Grove discussed this transformation, employing line and point coördinates, and showed how one curve can be obtained from the other. He discussed, in particular, the case in which the caustic is a circle with its center at the source of light, and established a geometric characteristic of the reflecting curves for this case.

3. *Sequences of Kummer bounds and modified continued fractions*, by Professor J. W. Bradshaw, University of Michigan.

Employing Kummer's method, Professor Bradshaw calculated a series of bounds for $\sum_{n+1}^{\infty} k^{-2}$ and $\sum_{n+1}^{\infty} k^{-3}$. He showed that they constitute "modified continued fractions" as defined in his article in this MONTHLY, vol. 49, 1942, pp. 513–519.

4. *A Cantor function constructed with continued fractions*, by Fritz Herzog and B. H. Bissinger, Michigan State College, introduced by the Secretary.

The authors define a function $w(x)$ on the interval $0<x<1$ as follows. If x has a simple continued fraction expansion $x=\{c_1+1, c_2+1, \cdots\}$ in which no c_i is zero, then $w(x)=\{c_1, c_2, \cdots\}$. Otherwise $w(x)$ has the same value as for the next such number less than x. Then $w(x)$ is continuous, non-decreasing, and constant almost everywhere, but is not absolutely continuous. Its graph is rectifiable with length 2. At each point of a set of the power of the continuum the two upper derivatives are equal, positive, and finite, while the two lower derivatives are zero. Thus the derivative fails to exist.

5. *Mathematics in Scotland before 1700*, by Professor E. R. Sleight, Albion College.

A considerable time spent in careful investigation on the site has enabled Professor Sleight to present a clear picture of the early development of mathematics in Scotland. He finds two periods, the first in which the study of mathematics was confined to the monasteries and restricted to the elements of arithmetic, algebra, and geometry; and the second in which, after the founding of the universities, a gradual development took place. This was, however, rather sporadic until the latter half of the seventeenth century.

6. *The summation of* $\sum 1/n$, by Professor C. J. Coe, University of Michigan.

Let a row of rectangles of unit width and with altitudes $1, 1/2, 1/3, \cdots$ be placed in contact on the x-axis. We seek a continuous curve $y=f(x)$ such that the area under the curve between the sides of each rectangle shall be equal to the area of the rectangle. In other words, it is required that

$$\int_{x-1}^{x} f(t)dt = \frac{1}{x}.$$

It was shown that a particular determination of $f(x)$ is the trigamma function

$$f(x) = \sum_{i=1}^{\infty} (x+i)^{-2} = \frac{d^2}{dx^2} \log (x!).$$

Various properties of the function were developed, and it was shown that it could be represented by any one of a sequence of more and more rapidly convergent series and also by certain asymptotic series.

7. *A cubic surface with rational lines*, by Professor J. S. Frame, Michigan State College.

Professor Frame discussed the determination of cubic surfaces having 27 distinct real lines with coördinates rational in some projective system. By choosing the coördinate planes and the unit plane as the tritangent planes to the surfaces, the equation in projective coördinates (x_1, x_2, x_3, x_4) becomes $x_0x_2x_4 = x_1x_3x_5$ where

$$x_0 + x_1 + x_2 + x_3 + x_4 = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0,$$

and in which $a_1, \cdots, a_5$ are homogeneous parameters. A plane $\rho x_0 = a_5x_5$ will be tritangent only when ρ is one of the roots λ, μ, ν of the cubic

$$\rho(\rho - a_2)(\rho - a_4) = a_5(\rho - a_1)(\rho - a_3).$$

If we let

$$\xi_0 = \rho x_0, \quad \xi_5 = -a_5x_5, \quad \xi_i = (\rho - a_i)x_i, \quad i = 1, 2, 3, 4$$

we may write the equation of the surface in the form

$$\xi_0\xi_2\xi_4 + \xi_1\xi_3\xi_5 = 0$$

or in any one of the 27 forms obtained from

$$\xi_0(\xi_2 + \xi_1)(\xi_4 + \xi_1) + \xi_1(\xi_3 + \xi_0)(\xi_5 + \xi_0) = 0$$

by letting ρ be λ, μ, ν, and permuting the subscripts 0, 2, 4 and 1, 3, 5 respectively. The first equation displays the nine lines of the form $\xi_0 = \xi_1$, and the second displays the eighteen lines of the form

$$\xi_0 + \xi_1 = \xi_2 + \xi_3 = \xi_4 + \xi_5 = 0.$$

For these lines to be rational we must have the a_i and λ, μ, ν all rational. To accomplish this, assign integral values to a_1, λ, μ, ν, compute the number

$$F = \frac{1}{a_1}(a_1 - \lambda)(a_1 - \mu)(a_1 - \nu) = (a_1 - a_2)(a_1 - a_4)$$

and let $a_2 - a_1$ be any factor R of F. Then

$$a_2 = a_1 + R, \quad a_4 = a_1 + \frac{F}{R}, \quad a_5 = \lambda + \mu + \nu - a_2 - a_4, \quad a_3 = \frac{\lambda\mu\nu}{a_1a_5}.$$

For the 27 lines to be distinct we exclude cases in which a_2 or a_4 is equal to λ, μ, ν. An example of a surface with 27 distinct rational lines is

$$(x_1 + x_2 + x_3 + x_4)x_2x_4 - x_1x_3(2x_1 + 3x_2 + 15x_3 + 8x_4).$$

8. *Sum of the distances to sides of a triangle*, by D. K. Kazarinoff, University of Michigan, introduced by the Secretary.

C. J. Coe, *Secretary*

THE FALL MEETING OF THE INDIANA SECTION

The twenty-third annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on Friday, October 19, 1945, in conjunction with the fall meeting of the Indiana Academy of Science. Professor Juna L. Beal presided.

Thirty-three persons registered at the meeting, including the following twenty-one members of the Association: G. E. Albert, W. L. Ayres, Juna L. Beal, Stanley Bolks, I. W. Burr, W. E. Edington, G. H. Graves, Cora B. Hennel, H. K. Hughes, M. W. Keller, E. L. Klinger, H. A. Meyer, A. N. Milgram, G. T. Miller, Ivan Niven, P. M. Pepper, C. K. Robbins, T. Y. Thomas, M. S. Webster, H. E. Wolfe.

At the business meeting the following officers were elected for the next year: Chairman, W. L. Ayres, Purdue University; Vice-Chairman, G. H. Graves, Purdue University; Secretary-Treasurer, M. W. Keller, Purdue University. It was decided to hold the next annual meeting again in conjunction with the Indiana Academy of Science.

The following papers were presented:

1. *A practical form of the comparison test for series of positive terms*, by Professor H. K. Hughes, Purdue University.

Let $\sum u_n$ and $\sum v_n$ be two infinite series of positive terms. If $\lim_{n\to\infty}(u_n/v_n)=L$, where L is a positive number, then it is well known that the two series are both convergent or both divergent. This modified form of the usual comparison test has not been much used in classes studying series for the first time, but it actually is very practical. The speaker cited examples of series for which a young student might have difficulty in setting up a "comparison series" but which could be handled easily by the test in the form here described.

2. *Statistical methods for controlling the quality of industrial products*, by Professor I. W. Burr, Purdue University.

Since industrial data are statistical in nature, it is only to be expected that they may best be analyzed by statistical methods. In this connection the concepts of frequency distribution, control charts, correlation, and probability are especially useful. It was the purpose of this paper to show how these tools are used in the practical applications, to suggest this field as a new and attractive career, and to point out that there are many unsolved problems.

3. *A program for increasing interest in mathematics in Indiana high schools*, by Professor W. H. Carnahan, Purdue University, introduced by M. W. Keller.

The speaker illustrated by various devices the manner in which he attempts to interest high school students in mathematics by showing them the part it plays in their daily lives.

4. *On length of curves*, by Professor A. N. Milgram, University of Notre Dame.

Let R be a plane region bounded by a simple closed curve J. We say that R bends toward the region at the point $P\subset J$ if there exist arbitrarily small segments in R with endpoints A and B on J in the order APB on a "small"sub arc of J. *Every simple closed curve has at least three points at which the curve bends toward the region.* This may be used to prove that in any closed region which is simply connected two interior points have a unique geodesic joining them. A

simple closed curve J of finite length L has in any ϵ neighborhood a curve interior to J of length $L^*<L$, and exterior to J a curve of length $L^*<L+\epsilon$.

5. *Absolute scalar invariants and the isometric correspondence of Riemann spaces*, by Professor T. Y. Thomas, Indiana University.

Necessary and sufficient conditions for the isometric correspondence of Riemann spaces R_n and $\overline{R}_n$ are given in terms of the equality of absolute scalar invariants of the spaces. In the general case for which the spaces admit a complete set of n functionally independent scalars, it is proved that these and a certain derived set of scalars suffice for the solution of the problem. The solution of the corresponding problem is given for spaces of two dimensions which do not admit two functionally independent scalars.

6. *Symmetry in metric spaces*, by Professor P. M. Pepper, University of Notre Dame.

In an abstract metric space S, a point c is called a *center of pointwise symmetry* if for each x in S there exists a point $y(x)$ such that the distance xc is equal to the distance $cy(x)$ and one half the distance $xy(x)$. If S has at least two centers of pointwise symmetry, then S is unbounded. A point c of S is called a *center of η-symmetry* (fractional symmetry) if $0<\eta\leqq 1$, and for each x in S there exists a point $y(x)$ for which $xc=cy(x)$ and $xy(x)\geqq 2\eta xc$. For each positive η less than 1 there exist bounded metric spaces of arbitrarily small diameter with two centers of η-symmetry. (Examples related to the Chebychef polynomials of the second kind are shown for each η less than 1.) A point c of S is called a *center of pointwise open symmetry* if for each number $\eta>0$ and each x in S there exists a point $y(\eta, x)$ such that $xy(\eta, x)\geq 2xc-\eta$ and $|cy(\eta, x)-xc|\leqq\eta$.

M. W. KELLER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Summer Meeting, Ithaca, New York, August 19–20, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Peoria, May 9-10, 1947
INDIANA, Terre Haute, October 18, 1946
IOWA
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LOUISIANA-MISSISSIPPI
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METROPOLITAN NEW YORK
MICHIGAN
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NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
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